PROBLEM SET 01
the GENESEAN CRISIS

Χαῖρε, friend. I'm surprised you've accepted the challenge, but am glad for it. I was thinking a bit about land and war and how things fit together. Tell me if any of this makes sense:

When I was a boy, there was a war between Herania and Hadestum. I grew up far away from there, but I still have vivid memories of the build-up to war. In particular, military officials in Herania seemed to have trouble figuring out just how to execute their invasion of Hadestum: west of Lake Heracles, a little out of the way but with a smooth path forward? East of Lake Hippolyta, more direct from Argos but through the rugged mountains? Between the two on the Isthmus of the Belt, with its abundant resources and industrial capacity but heavily armed and in a strategic bottleneck?

In a sense, though, this quandary had the benefit of leaving Hadestan commanders confused about how best to defend! Confusion—it's contagious.

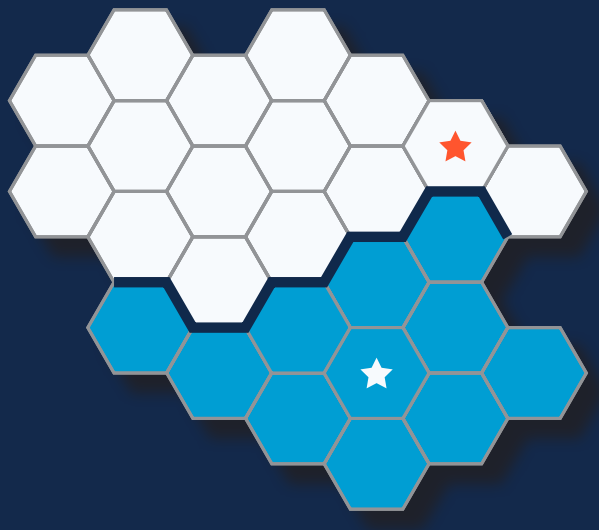
SocrateS

Geneseo is a pretty nice place, so it stands to reason that its two states find themselves embroiled in the occasional territorial conflict. Three glacial seas carve the landmass into several twisty and turny regions; these seas and the land around them are of particular strategic value, both militarily and economically. A newly-constructed canal linking the lakes has only enhanced this strategic value.

[*We leave aside complicated intrastate matters in Hadestum, where the island region of Persephone enjoys certain political benefits, such as their own parliamentary body. Persephes speak their own unique dialect of Neoclassical Hadesti that mixes Proto-Hadesti, indigenous languages like Vulgar Rhean, and the Received Heranza—the last of these serving as long-lasting residue of Heranian occupation several hundred years ago.*]

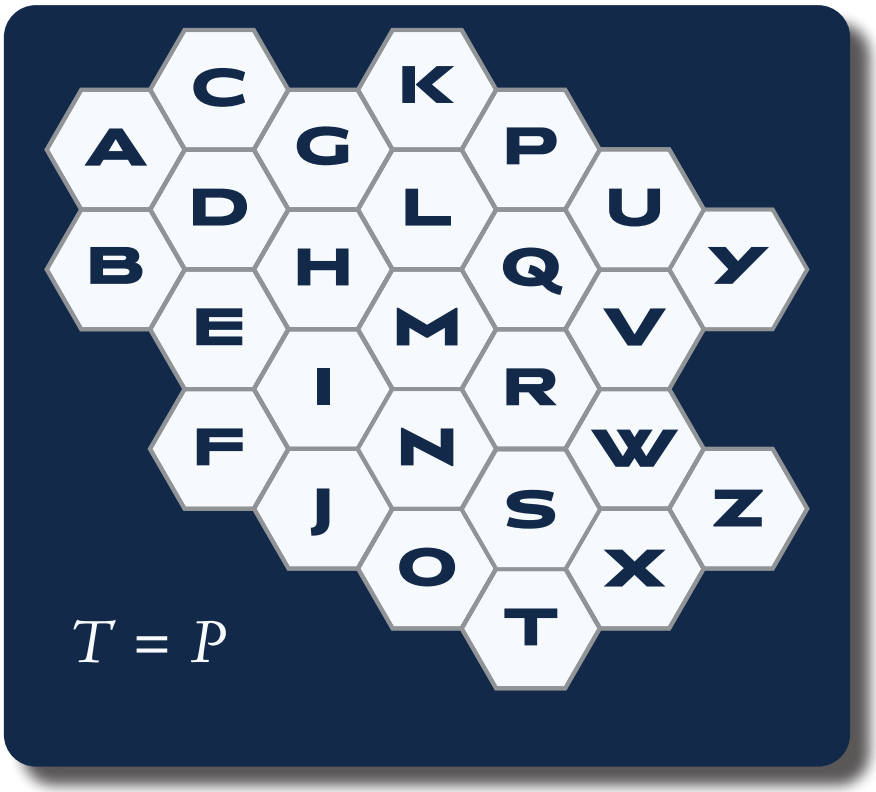
Land played a big part in the previous war between Herania and Hadestum. I was wondering if it was possible to think hard about land. We need to simplify the picture a bit, so I'll superimpose this incredibly-original, never-before-seen hexagonal grid overtop the map. This divides Geneseo into 26 little hexes.

At the moment, Herania occupies the northern 15 hexes, while Geneso occupies the southern 11. Herania's capital, Argos, lies to its extreme east. Hadestum's capital Tartarus lies close to its center, on the edges of the fertile Asphodel Meadows.

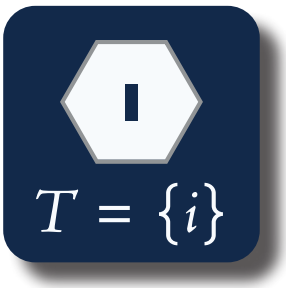


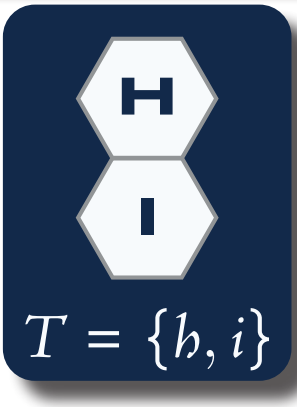
Maybe we can practice thinking on this map to better understand a state's territorial animations and, what is more, how those animations maps to the choices states make. So, let us (momentarily) rid the map of its current political details and make it our little plaything.

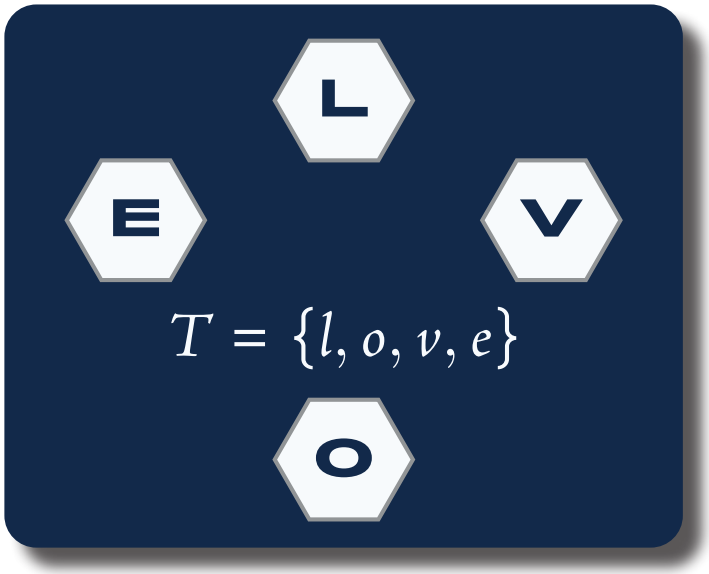
PART 1 | A MATTER OF LAND

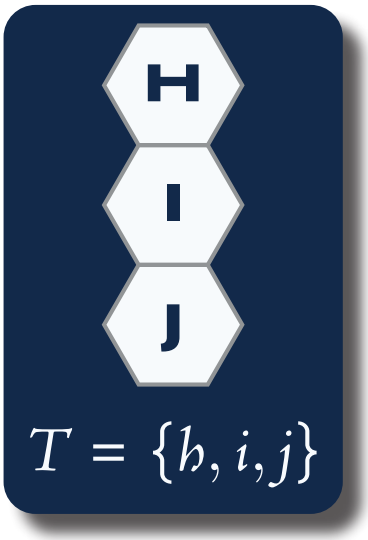


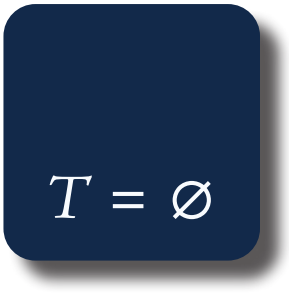
Let us begin by thinking about preferences over territorial pieces. We'll call each of these hexes a *parcel*. The set of all parcels is easy to write: $P = \{a, \ldots, z\}$. By a *territory*, we simply mean a subset of the set of parcels, so that if we write $T \subseteq P$, we mean we're talking about some territory named T which contains things from P —which is to say, some parcels. Here are some territories for your edification:











And I want to come up with a rule to help me see what happens under the maxim "more land is better than less." To that end, I think we should write in our strange symbols, just to be sure things work properly. I think it'll be easiest if we start with the preference rule

$$T_1 \succcurlyeq_{count} T_2 \iff |T_1| \geqslant |T_2|,$$

where by $|T|$ I mean the number of parcels contained in a territory T. So, this rule rewards territories that cover many parcels, irrespective of parcel characteristics.

For example, maybe we have two territories called $T_1 = \{i, l, n\}$ and $T_2 = \{b, o, i, l, e, r, m, a, k, s\}$. (From what magical place am I conjuring these sets?!) Then we have $|T_1| = 3$ and $|T_2| = 10$, implying $T_2 \succ T_1$.

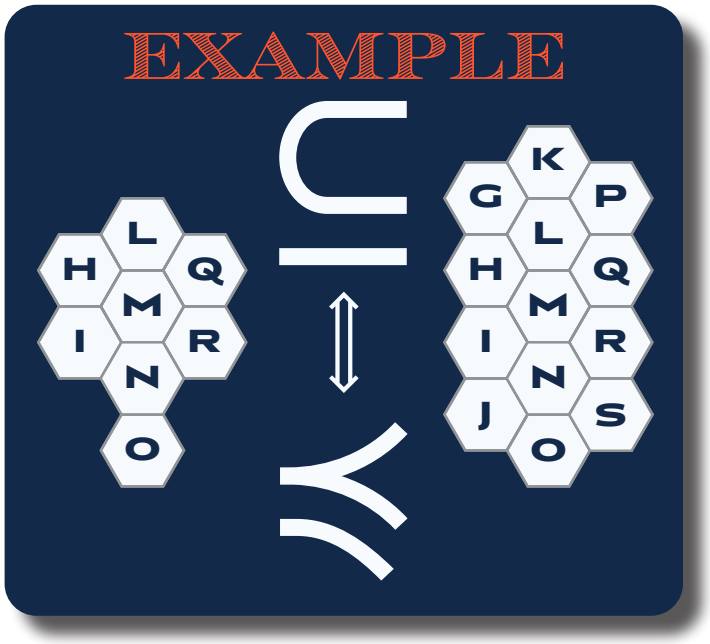
1. Use the rule to construct the most precise possible statement comparing $\{l, o, v, e\}$ and $\{h, a, t, e\}$.
2. Use the rule to construct the most precise possible statement comparing $\{l, o, v, e\}$ and $\{c, l, o, v, e, r\}$.
3. Is this rule reflexive? Why or why not?
4. Is this rule complete? Why or why not?
5. Is this rule transitive? Why or why not?
6. Characterize the associated strict preference and indifference rules.
7. Does there exist a possible territory strictly preferred to all other possible territories?
8. Does there exist a possible territory that is strictly worse than all other possible territories?
9. If the rule is regular, construct a utility function whose domain is $\{T \mid T \subseteq P\}$ and whose codomain is $\{0, \ldots, 26\}$. If the rule is not regular, construct a minimal example showing where the utility function fails to work properly.

10. Use the rule to construct the most precise possible statement comparing $\{l, o, v, e\}$ and $\{h, a, t, e\}$.
11. Use the rule to construct the most precise possible statement comparing $\{l, o, v, e\}$ and $\{c, l, o, v, e, r\}$.
12. Is this rule reflexive? Why or why not?
13. Is this rule complete? Why or why not?
14. Is this rule transitive? Why or why not?
15. Characterize the associated strict preference and indifference rules.
16. Does there exist a possible territory strictly preferred to all other possible territories?
17. Does there exist a possible territory that is strictly worse than all other possible territories?
18. If the rule is regular, construct a utility function whose domain is $\{T \mid T \subseteq P\}$ and whose codomain is $\{0, \ldots, 26\}$. If the rule is not regular, construct a minimal example showing where the utility function fails to work properly.
19. Is it the case that
$$T_1 \succcurlyeq_{count} T_2 \implies T_1 \succcurlyeq_{lazy} T_2?$$
20. Is it the case that
$$T_1 \succcurlyeq_{lazy} T_2 \implies T_1 \succcurlyeq_{count} T_2?$$

This part is worth 4 points; each question is worth 0.20 points.

But, it doesn't seem very likely to me that all of the parcels are equally valuable. We were just talking about how certain parts of Geneseo were particularly important! But, I am not in the mood to go look up which parcels are valuable and which are not. That sounds hard. Instead, let me be a bit lazy and try the rule

$$T_1 \succcurlyeq_{lazy} T_2 \iff T_2 \subseteq T_1.$$



yes, this is hard. but *you can do it.*

PART 2 | LOOK IT UP

OK, OK, we can do better. I looked it up. Suppose that we now know which parcels are more valuable than others. I will write this as follows:

$$\underbrace{p_1 \trianglerighteq_i p_2}.$$

In the judgment of State i,
p_1 is at least as valuable as p_2.

For example, the Hadestans' favorite parcel is clearly the one that contains Tartarus, s. Similarly, the Heranians' favorite parcel is the one that contains Argos, u. Both of them agree that parcel m is good, though clearly not as important as defense of the capital.

Let us call the number of parcel-steps it takes to get from parcel x to parcel y $d(x, y)$. For our two states, we define preference rules that reward proximity to capital cities and to the strategic stronghold:

$$p_1 \trianglerighteq_{\text{Hera}} p_2 \iff 2d(u, p_1) + d(m, p_1) \leqslant 2d(u, p_2) + d(m, p_2),$$
$$p_1 \trianglerighteq_{\text{Hades}} p_2 \iff 2d(s, p_1) + d(m, p_1) \leqslant 2d(s, p_2) + d(m, p_2).$$

1. Draw a preference doodle for Herania.
2. Draw a preference doodle for Hadestum.
3. Are these rules reflexive? Why or why not?
4. Are these rules complete? Why or why not?
5. Are these rules transitive? Why or why not?
6. If the rules are regular, construct a utility function for both states. If they are not regular, construct a minimal example showing where the utility function fails to work properly.

This part is worth 3 points; each question is worth 0.25 points.

[*Hint:* have questions, and don't freak out: these are all easier than they look, and you're plenty clever to handle the wrinkles. Most of all, have fun.]

Now, can we take advantage of the new information we've gotten about parcels to construct a better rule about territories? Let's see. Suppose that I constructed a utility function over parcels determined by each state's preference relation as studied at left: $u_i : P \to \mathbb{R}$, where the idea is that we have

$$p_1 \trianglerighteq_i p_2 \iff u_i(p_1) \geqslant u_i(p_2).$$

Now I'll try to use that to learn about territories:

$$T_1 \succcurlyeq_i T_2 \iff \sum_{p \in T_1} u_i(p) \geqslant \sum_{p \in T_2} u_i(p).$$

7. Is this rule reflexive? Why or why not?
8. Is this rule complete? Why or why not?
9. Is this rule transitive? Why or why not?
10. Is it the case that
$$T_1 \succcurlyeq_{count} T_2 \implies T_1 \succcurlyeq_i T_2?$$
11. Is it the case that
$$T_1 \succcurlyeq_i T_2 \implies T_1 \succcurlyeq_{count} T_2?$$
12. Is it the case that
$$T_1 \succcurlyeq_{lazy} T_2 \implies T_1 \succcurlyeq_i T_2?$$
13. Is it the case that
$$T_1 \succcurlyeq_i T_2 \implies T_1 \succcurlyeq_{lazy} T_2?$$

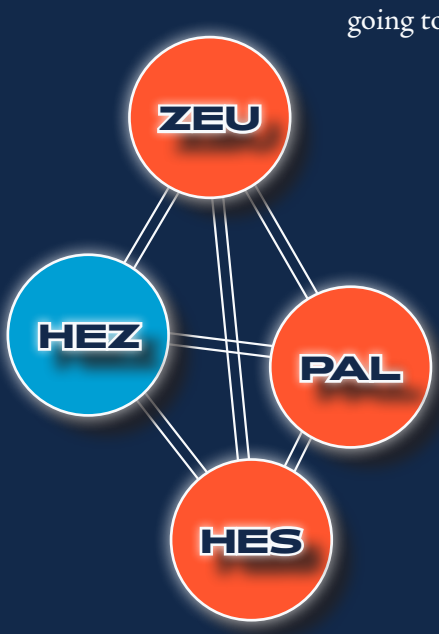

PolitaN OceaN
Agonus
N'ZeusK
Caduceus
The HermesiatiCs
Lemnos
The HephaestineS
AphroditwE
EroS
TragiC OceaN
AreshU
PhobOS
DeimOS
Bulo
PallassA
The PrytaneioN
HestalluM
The Olympian Prytaneion
CronaN OceaN
In the
Olympian Prytaneion...
Meanwhile
Phœbustan
Daphne
PoH-Sai-DaN
Enosighthon
CronaN OceaN
HeraniA
Ares
Tartarus
HadestuM
GaiaN OceaN
Orion
ArtemissiA
DionyS
Tinnus

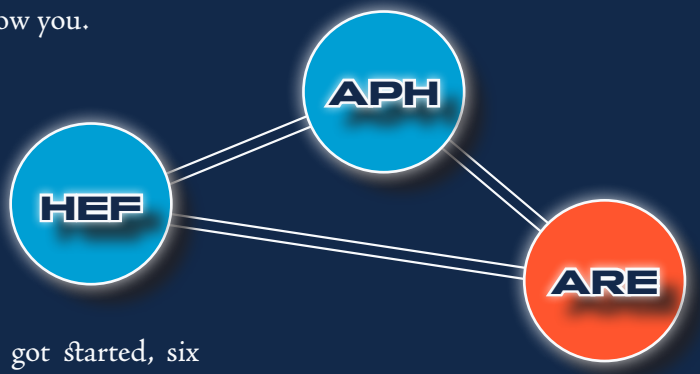
PART 3 | INTERVENTION

So, how to solve this problem?

After Chaos IV, all thirteen states of Olympia agreed to develop the Olympian Union: an international institution tasked not only with staving off world wars before they started, but also with bringing positive peace and prosperity to the world. Being the only sovereign state to stay out of the Chaos, Hestallum seemed like a natural fit to serve as home of the Union. After some bickering about whether one state should house the union, the clever Hephæstans suggested that the world's problem be solved the same way we Hephæstans handled it hundreds of years ago: by building a new island distinct from all other political bodies! But whereas our capital, Lemnos, stands proudly as a symbol of human ingenuity, the Olympian Prytaneion, well, sucks. It sucks real bad.

It sucks because it never got finished! And I think it never got finished because we wrote down stupid rules for how things ought to go there! And I think that if we're not careful, it's all going to happen again! Let me show you.

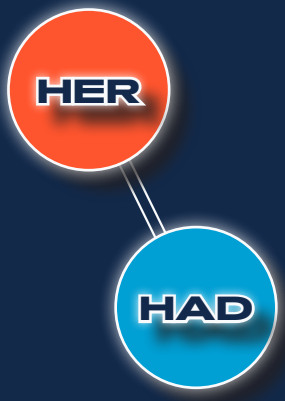




When the Union first got started, six states—n'Zeusk, Pallassa, Hestallum, Herania, Areshu, and Artemissia—were named "Special Members," but we all knew that meant that meant: they were six of the most powerful states with the most to lose from the loss of sovereignty. The other seven states, while afforded the right to vote in the Union, were afforded fewer institutional benefits.

Naturally, the relatively-weak Hestallans found their way into Special Status, which I *think* is because they're close allies with Pallassa and Herania. Those jerks always get what they want!

[*Or, maybe it's because nearly all diplomats speak Simplified Hestallan? But I feel like that might be endogenous...*]



The Compact of the Olympian Union established the following voting rule.

Let $O = \{1, \ldots, 13\}$ be the set of Olympian states, and let $S = \{1, \ldots, 6\}$ be the set of Special Members.

For each vote, a given state can vote one of three ways: Yes, No, or Concur. The idea is that a state can express a strong preference with a Yes or a No vote, but they also can say "I won't stand in anybody's way" with a Concur vote. After all, you can irritate enemies with a No vote just as much as you can with a Yes!

We assume the Union's rule is complete—how can you govern if you don't issue decisions?!—& encode the Union's decision rule as follows:

CODE

$x \succ_i y \rightsquigarrow$ State i votes Yes

$x \succsim_i y \rightsquigarrow$ State i votes Yes or Concur

$y \succ_i x \rightsquigarrow$ State i votes No

CODE

$$\underbrace{x \succ y \iff}_{\text{The Union strictly prefers } x \text{ to } y \text{ if, and only if...}} \overbrace{\left|\{i \in O \mid x \succ_i y\}\right| \geqslant 8}^{\text{...at least 8 states vote Yes, and...}} \wedge \underbrace{(\forall j \in S)\left[x \succsim_j y\right]}_{\text{...all Special Members vote Yes or Concur.}}$$

Help me think this through:

1. Does the Union's rule satisfy Weak Pareto?
2. Does the Union's rule satisfy IIA?
3. Does the Union's rule satisfy No Dictators?
4. Does the Union's rule satisfy Transitivity?
5. Is the "strict part" of the Union's rule transitive?
6. Is the "indifference part" of the Union's rule transitive?
7. Write out the set of all coalitions of states with the following property: if all states in the coalition vote Yes, then their preferred alternative is strictly preferred by the Union.

This part is worth 3 points; each question is worth 0.5 points.

I find tables like this one very helpful when ruminating about the Olympian Union!

z	y	x	ZEU	01
	xy	z	PAL	02
yz		x	HES	03
	y	xz	HER	04
x	y	z	ARE	05
z		xy	ART	06
	xz	y	HEZ	07
x		yz	HAD	08
x	z	y	HEF	09
x	z	y	APH	10
z	x	y	APO	11
	xyz		POH	12
y	x	z	DIO	13
			UNION	

→ *better to the right*

also, i feel really sheepish about this, but...i need your help with something...

I will be visiting my sister Adelphia and niece Sophia! They live just a quick flight away in Bubo, the capital of Pallassa. I love visiting Pallassa, since it has so many wonderful universities. Well, at least East Pallassa does. West Pallassa has always been a bit behind, perhaps because of all the people who join the Medusaheads—that's a nickname for the Pallassan elite guard. But you already knew that.

I haven't seen Sophia since she was 7, and now she nearing age 10. I spend too much time in my stupid office on that stupid half-island. I am so excited to see her...but, I don't quite know what to get her as a sorry-I've-been-such-a-bad-uncle present. I've been reading some blogs about what books to buy for percocious girls like her, but I'm still on the fence.

This is so embarrassing. All of the blogs I read told me that I should get Sophia one of the books from the *Ioanna Patsasoglou* series. But there are so many of them! The blogs all said that some of them were really good, some were good, some were just OK, some were bad, and some were really bad. But none of the blogs say the same thing about them!

Of course, as I am a good scholar, I have compiled data prior to making my purchase. Surely Sophia will be impressed by my efforts. What's that you say? Kids don't care how hard you work to get a present? Balderdash. Though you may be on to something...

Anyway, here are the data I compiled about the books. Each entry is the proportion of blogs that said a given book (rows) is of a given quality (columns). I think we should give each kind of book a score: really good books get a score of 2; good books get a score of 1; meh books get a score of 0; bad books get a score of −1; and really bad books get a score of −2.

Oh! I need to get to my flight! Can you do the calculations and text me the results so I can buy the right book at the airport? kthxbai

PART 4 | WHICH BOOK?

BOOK	VERY GOOD	GOOD	MEH	BAD	VERY BAD
Ioanna Patsasoglou and the NECROMANCER'S CHOLELITH	0.0	0.0	0.2	0.4	0.4
Ioanna Patsasoglou and the ENCLAVE OF KASSANDRA	0.1	0.0	0.0	0.0	0.9
Ioanna Patsasoglou and the CONVICTED CYCLOPS	0.0	0.5	0.3	0.1	0.1
Ioanna Patsasoglou and the INSULATED DISKOPOTIRO	0.3	0.0	0.0	0.6	0.1
Ioanna Patsasoglou and the BENNU TAXIS	0.2	0.3	0.5	0.0	0.0
Ioanna Patsasoglou and the HIRSUTE COUSIN	0.1	0.0	0.0	0.5	0.4
Ioanna Patsasoglou and the PRAYER THAT BACKFIRED	0.1	0.4	0.4	0.1	0.0

This shouldn't be too hard, but that's easy for me to say.

1. Using the utility scores mentioned on the previous page, compute the expected utility for each of the seven books.
2. Using the expected utilities you computed, draw a preference doodle for the books, where there is an arrow from one book to another book if and only if the first book has an expected utility at least as high as the second.
3. Which of the books are better (in expectation) than a book that's Meh for sure?
4. Suppose now that I wanted to avoid looking like an idiot, so that now a Very Bad rating yields −5 happiness points instead of −2. Does the ranking change?

this question is worth 5 points:

0.5 for each of the expected utilities, for a total of 3.5 available there, plus 0.5 for the three remaining questions.