

Sequences and limits

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1 Motivation

Many of the things political economists want to say about a process unfolding over time — a sequence of polling estimates approaching an unknown population mean, a sequence of best responses approaching a Nash equilibrium, a sequence of price adjustments approaching a market-clearing price, a sequence of cross-country growth rates approaching a steady state — are claims about *limits*. The numerical sequence keeps moving and never literally arrives, but for any standard of closeness one cares to specify, the sequence is eventually that close to its limit. The basic move of analysis is to make this precise: what does it mean to say a sequence approaches a limit, and what kinds of sequences are guaranteed to do so?

The order-theoretic groundwork is already in place. The least upper bound property of \mathbb{R} — every nonempty bounded-above subset has a supremum — is the structural fact that distinguishes \mathbb{R} from \mathbb{Q} and that makes the whole machinery of limits work. (The rationals do not have it, and the reals are essentially defined to be the smallest extension of \mathbb{Q} that does.) From the LUB property follow the three workhorse completeness theorems of real analysis: monotone convergence, Cauchy completeness, and the Bolzano–Weierstrass theorem on subsequences. This handout develops the basics of convergence, gives the LUB property a section of its own (with the construction of \mathbb{R} from \mathbb{Q} as an extended footnote), and walks through the completeness triangle. It closes with \limsup and \liminf , the right tools for sequences that don’t converge but are still bounded.

2 Sequences and convergence

A sequence is the simplest model of a process unfolding in time. Every monthly polling average, every quarterly GDP figure, every year’s election turnout, every iteration of a tatonnement adjustment, every step of a learning algorithm — read in order — is a sequence of real numbers, and the question of whether such a sequence settles down to a stable target is the prototypical analysis question. The ϵ - N formulation in this section is the formal capture of “settles down,” and is the move on which everything else in the analysis cluster builds.

Formally, a sequence is a list of real numbers indexed by the natural numbers. The standard notation is $(a_n)_{n \in \mathbb{N}}$, or just (a_n) when the index set is understood. This is the same thing as a function $a : \mathbb{N} \rightarrow \mathbb{R}$, with a_n shorthand for $a(n)$. We will not always start the indexing at 0 — $(a_n)_{n \geq 1}$ is also a sequence — and we will sometimes write out the first few terms $(1, 1/2, 1/3, \dots)$ when the pattern is more vivid that way.

The basic adjectives. A sequence (a_n) is *bounded* if there exists $M \in \mathbb{R}$ with $|a_n| \leq M$ for every n ; equivalently, the set $\{a_n : n \in \mathbb{N}\}$ is bounded in \mathbb{R} . It is *monotone increasing* if $a_n \leq a_{n+1}$ for every n , *strictly increasing* if the inequality is strict, and *monotone decreasing* / *strictly decreasing* dually. “Eventually” is shorthand for “for all n from some point onward”: (a_n) is *eventually positive* if there

exists N with $a_n > 0$ for every $n \geq N$, and similarly for any other property. The eventual behavior of a sequence is what matters for limits; the early terms can be anything.

The central definition of analysis is the following.

Definition 1. A sequence (a_n) *converges* to a real number L , written $a_n \rightarrow L$ or $\lim_{n \rightarrow \infty} a_n = L$, if for every $\epsilon > 0$ there exists a natural number N such that

$$|a_n - L| < \epsilon \quad \text{for every } n \geq N.$$

A sequence is *convergent* if it converges to some L , and *divergent* otherwise.

The structure of the definition is worth pausing on.¹ The variable ϵ ranges over arbitrary positive reals, the threshold for what counts as “close.” The number N is allowed to depend on ϵ : the smaller the ϵ , the larger N may need to be. What the definition guarantees is that for any standard of closeness, the sequence is eventually within that standard.

Example 2 (The basic shrinking sequence). $1/n \rightarrow 0$. Given $\epsilon > 0$, by the Archimedean property of \mathbb{R} there is some $N \in \mathbb{N}$ with $1/N < \epsilon$. Then for every $n \geq N$, $|1/n - 0| = 1/n \leq 1/N < \epsilon$.

Example 3 (Constant and eventually-constant sequences). The constant sequence $a_n = c$ converges to c : for any $\epsilon > 0$, take $N = 0$. More generally, any eventually constant sequence ($a_n = c$ for $n \geq N_0$) converges to c with $N = N_0$.

Example 4 (An ϵ - N argument with arithmetic). $a_n = n/(n+1) \rightarrow 1$. Compute $|a_n - 1| = 1/(n+1)$. Given $\epsilon > 0$, choose N with $1/(N+1) < \epsilon$ (any $N \geq 1/\epsilon$ works). Then $n \geq N$ gives $|a_n - 1| = 1/(n+1) \leq 1/(N+1) < \epsilon$.

The limit, when it exists, is unique.

Proposition 5. *If $a_n \rightarrow L$ and $a_n \rightarrow L'$, then $L = L'$.*

Proof. Suppose for contradiction $L \neq L'$, and let $\epsilon = |L - L'|/2 > 0$. Choose N_1 such that $|a_n - L| < \epsilon$ for $n \geq N_1$, and N_2 such that $|a_n - L'| < \epsilon$ for $n \geq N_2$. For $n \geq \max(N_1, N_2)$,

$$|L - L'| \leq |L - a_n| + |a_n - L'| < 2\epsilon = |L - L'|,$$

a contradiction. □

The triangle inequality argument $|x - y| \leq |x - z| + |z - y|$ used here, and a few small variants, are the staple moves of ϵ - N proofs. The next proposition assembles the basic algebra of limits with the same machinery.

Proposition 6 (Algebra of limits). *Suppose $a_n \rightarrow L$ and $b_n \rightarrow M$. Then:*

¹The ϵ - N formulation crystallized in the second half of the nineteenth century, in the work of Bolzano, Cauchy, and Weierstrass, replacing earlier infinitesimal-based formulations that had served well in the calculus of Newton and Leibniz but proved hard to make rigorous as the demands on analysis grew. The intuitive content is the same on both formulations — “ a_n gets arbitrarily close to L ”. The rigorous content lives in the order of quantifiers: the choice of N is allowed to depend on ϵ , but ϵ is universally quantified *first*. Reversing the quantifiers — “there is an N such that for every $\epsilon \dots$ ” — yields a strictly stronger condition (the sequence is eventually exactly equal to L) that is rarely what one wants. Several places in analysis turn on similar quantifier subtleties: pointwise versus uniform convergence of functions, continuity versus uniform continuity, integrability versus uniform integrability. Getting the quantifier order right is most of what makes a definition in analysis the right one.

1. $a_n + b_n \rightarrow L + M$;
2. $a_n b_n \rightarrow LM$;
3. if $M \neq 0$ and $b_n \neq 0$ for all n , then $a_n/b_n \rightarrow L/M$.

Convergent sequences are bounded.

We omit the proofs — they are standard ϵ - N exercises following the same triangle-inequality pattern as Proposition 5, and the reader who has not seen them before should write at least the sum and product cases out by hand.

A *subsequence* of (a_n) is a sequence $(a_{n_k})_{k \in \mathbb{N}}$ where $n_0 < n_1 < n_2 < \dots$ are natural numbers — intuitively, “pick out the entries at a subsequence of indices, preserving order.” Convergence is hereditary in the natural way:

Proposition 7. *If $a_n \rightarrow L$, then every subsequence (a_{n_k}) also converges to L .*

Proof. Given $\epsilon > 0$, choose N as in the definition of $a_n \rightarrow L$. Since $n_k \geq k$ (a strictly increasing sequence of natural numbers grows at least linearly), $k \geq N$ implies $n_k \geq N$, and so $|a_{n_k} - L| < \epsilon$. \square

The contrapositive is the more practical form: if a sequence has two subsequences converging to different limits, the sequence itself does not converge. The sequence $(-1)^n = -1, 1, -1, 1, \dots$ has even-indexed subsequence converging to 1 and odd-indexed subsequence converging to -1 , so $(-1)^n$ does not converge.

3 The least upper bound property

Why bother with \mathbb{R} rather than \mathbb{Q} ? Almost everything we measure in applied work is a rational number — vote shares, percentages, prices to the nearest cent, polling figures to the nearest tenth of a point. We never literally observe an irrational. Yet analysis is built on \mathbb{R} , and almost every theorem we will use takes the reals as the home of the action. The reason is structural: \mathbb{R} has the property that every nonempty bounded-above subset has a least upper bound, while \mathbb{Q} does not. (The set of rationals whose square is at most 2 is bounded above in \mathbb{Q} but has no least rational upper bound — there are smaller and smaller rationals greater than $\sqrt{2}$, with no smallest one in \mathbb{Q} .) This least-upper-bound property is the order-theoretic backbone of the convergence machinery, and it is what we cash in below to prove the workhorse completeness theorems.

Axiom 8 (Least upper bound property). Every nonempty subset of \mathbb{R} that is bounded above has a least upper bound (supremum) in \mathbb{R} .

By the duality between sup and inf (replace S with $-S$), the LUB property is equivalent to its lower-bound counterpart: every nonempty subset bounded below has a greatest lower bound (infimum). We assert the LUB property as an axiom because we are not constructing \mathbb{R} from \mathbb{Q} here — we are taking \mathbb{R} as given, with a real-valued ordering and the LUB property as part of the data.²

²There are two standard ways to construct \mathbb{R} from \mathbb{Q} , both due in essential outline to the late nineteenth century. *Dedekind cuts*: a real number is identified with a partition of \mathbb{Q} into a “lower” part (downward-closed, no greatest element) and an “upper” part. The order-theoretic content of the LUB property is built into this construction by

The LUB property fixes the gap at $\sqrt{2}$, and at every other irrational point besides; this is what we mean by saying that \mathbb{R} is *order-complete*.

The LUB property has an immediate consequence that is the workhorse of the next section.

Theorem 9 (Monotone convergence). *A monotone increasing sequence (a_n) in \mathbb{R} that is bounded above converges to $\sup_n a_n$. Dually, a monotone decreasing sequence bounded below converges to $\inf_n a_n$.*

Proof. Let $L = \sup\{a_n : n \in \mathbb{N}\}$, which exists by the LUB property. Given $\epsilon > 0$, $L - \epsilon$ is not an upper bound of the sequence (it is strictly less than the least upper bound), so there is some N with $a_N > L - \epsilon$. By monotonicity, for every $n \geq N$, $a_n \geq a_N > L - \epsilon$, and we always have $a_n \leq L < L + \epsilon$. So $|a_n - L| < \epsilon$ for $n \geq N$. The decreasing case is analogous. \square

The dependence runs both ways: monotone convergence implies the LUB property, and so the two are equivalent over the ordered-field axioms. We will return to this in the next section, where it joins Cauchy completeness in the completeness triangle.

4 Cauchy sequences and Bolzano–Weierstrass

Suppose you have been watching a sequence for a while and the consecutive entries are barely changing — the iterates of a learning algorithm have stopped moving by much, the values of a polling estimator have stabilized within a fraction of a percentage point — but you have no idea what number the sequence is converging to. Can you say it is converging anyway? In \mathbb{R} the answer is yes: that is Cauchy’s criterion, and it is the working analyst’s go-to convergence test when the candidate limit is unknown. The Bolzano–Weierstrass theorem is the same completeness fact wearing different clothes: every bounded sequence has at least one convergent piece. Together with the monotone convergence theorem of the previous section, these are the three most-used formulations of the structural fact that \mathbb{R} is complete — and we will see that all three are equivalent over the ordered-field axioms.

We have proved monotone convergence. Cauchy sequences are next.

Definition 10. A sequence (a_n) is *Cauchy* if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$|a_n - a_m| < \epsilon \quad \text{for every } n, m \geq N.$$

The Cauchy condition is ϵ - N convergence with the limit replaced by “each other”: not “the terms get close to L ,” but “the terms get close to each other.” The condition is intrinsic to the sequence (no candidate limit needs to be specified), which is the practical advantage when one suspects convergence but does not know what to.

design — the supremum of a bounded set of cuts is the union of their lower parts, which is itself a cut. *Cauchy completion:* a real number is identified with an equivalence class of Cauchy sequences of rationals (where two sequences are equivalent if their pointwise difference converges to 0). This construction makes Cauchy completeness obvious by design and gives a more topological flavor; the LUB property requires an argument but is straightforward. Both constructions yield the same \mathbb{R} up to order isomorphism (one can prove this either by checking the universal properties or by noting that any complete totally ordered field with a countable dense subset is unique up to isomorphism). Cauchy completion generalizes more cleanly to arbitrary metric spaces, which is why it is often preferred in modern treatments. Either way, what one is doing is filling in the “holes” in \mathbb{Q} where the LUB property fails — $\sqrt{2}$ being the iconic missing supremum — and the result is the smallest order-complete extension of \mathbb{Q} containing all the limits the rationals couldn’t reach. Rudin (1976) works through Dedekind cuts in detail; Folland (1999) and most graduate analysis texts use the Cauchy approach.

Proposition 11. *Every convergent sequence in \mathbb{R} is Cauchy.*

Proof. Suppose $a_n \rightarrow L$. Given $\epsilon > 0$, choose N with $|a_n - L| < \epsilon/2$ for $n \geq N$. For $n, m \geq N$,

$$|a_n - a_m| \leq |a_n - L| + |L - a_m| < \epsilon/2 + \epsilon/2 = \epsilon. \quad \square$$

The converse — every Cauchy sequence in \mathbb{R} converges — is the deeper direction and is one of the core completeness statements about \mathbb{R} .

Theorem 12 (Cauchy completeness of \mathbb{R}). *Every Cauchy sequence in \mathbb{R} converges.*

Proof sketch. A Cauchy sequence (a_n) is bounded: with $\epsilon = 1$, choose N with $|a_n - a_N| < 1$ for $n \geq N$, so $|a_n| \leq \max(|a_0|, \dots, |a_{N-1}|, |a_N| + 1)$. By Bolzano–Weierstrass (Theorem 13 below), (a_n) has a convergent subsequence $a_{n_k} \rightarrow L$. To show $a_n \rightarrow L$: given $\epsilon > 0$, choose N_1 such that $|a_n - a_m| < \epsilon/2$ for $n, m \geq N_1$, and choose K such that $|a_{n_k} - L| < \epsilon/2$ for $k \geq K$. Pick any $k \geq K$ with $n_k \geq N_1$ (possible since $n_k \rightarrow \infty$). For $n \geq N_1$, $|a_n - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \epsilon$. \square

The proof routes through Bolzano–Weierstrass. We owe a proof of that.

Theorem 13 (Bolzano–Weierstrass). *Every bounded sequence in \mathbb{R} has a convergent subsequence.*

Proof. Let (a_n) be bounded, with $a_n \in [c, d]$ for all n . Bisect $[c, d]$ into two halves; at least one half contains a_n for infinitely many n . Let $[c_1, d_1]$ be such a half, and pick n_1 with $a_{n_1} \in [c_1, d_1]$. Bisect $[c_1, d_1]$; pick a half $[c_2, d_2]$ containing a_n for infinitely many n greater than n_1 , and pick $n_2 > n_1$ with $a_{n_2} \in [c_2, d_2]$. Iterate. The intervals $[c_k, d_k]$ are nested with $d_k - c_k = (d - c)/2^k \rightarrow 0$. The sequence (c_k) is monotone increasing and bounded above by d , so by monotone convergence converges to some L ; (d_k) converges to the same L . Since $a_{n_k} \in [c_k, d_k]$, by the squeeze (between two sequences converging to L), $a_{n_k} \rightarrow L$. \square

These three results are the completeness triangle:

LUB property	Every bounded-above subset has a supremum.
Monotone convergence	Every bounded monotone sequence converges.
Cauchy completeness	Every Cauchy sequence converges.
Bolzano–Weierstrass	Every bounded sequence has a convergent subsequence.

Each of these implies the others over the ordered-field axioms.³ For applied work the practical takeaway is that any of the four can be invoked when handy — a monotone bounded sequence

³To sketch the equivalences: LUB \Rightarrow monotone convergence is Theorem 9. Monotone convergence \Rightarrow Bolzano–Weierstrass: a bounded sequence has a monotone subsequence (the standard “peak point” lemma — either there are infinitely many indices n with $a_n \geq a_m$ for all $m > n$, giving a monotone decreasing subsequence; or only finitely many, in which case after the last peak one can build a monotone increasing subsequence). The monotone subsequence is bounded, hence converges. Bolzano–Weierstrass \Rightarrow Cauchy completeness is Theorem 12. Cauchy completeness \Rightarrow LUB: given a nonempty bounded-above $S \subseteq \mathbb{R}$, build a Cauchy sequence of rational approximate suprema and take its limit. The first three implications are short; the closing one to LUB is the slowest. The point of stating the equivalence is conceptual: “completeness” is one structural property of \mathbb{R} , expressible in several languages. Different formulations are easier to apply in different settings — monotone convergence for series with positive terms, Cauchy completeness when no candidate limit is known, Bolzano–Weierstrass for compactness arguments. They are all the same structural fact wearing different clothes.

converges, a Cauchy sequence converges, a bounded sequence has a convergent subsequence, a bounded-above set has a supremum. They are interchangeable as tools.

The Cauchy formulation has one further conceptual virtue: it makes precise what fails in \mathbb{Q} . The sequence of decimal truncations of $\sqrt{2}$ — 1, 1.4, 1.41, 1.414, ... — is Cauchy in \mathbb{Q} (the differences shrink geometrically), but it does not converge in \mathbb{Q} , because its limit $\sqrt{2}$ is irrational. So $(\mathbb{Q}, |\cdot|)$ is *not* Cauchy-complete. The reals are precisely what one gets by adjoining limits of all such Cauchy sequences — which is the Cauchy-completion construction outlined in the §3 footnote.

5 lim sup and lim inf

Lots of empirical sequences in political science do not converge. Approval ratings drift up and down with no apparent destination; polling estimators bounce around between cycles; election returns swing between parties from one cycle to the next. For sequences like these, the limit does not exist — but two weaker quantities always do, and they capture much of what one cared about anyway. The *limit superior* and *limit inferior* are the upper and lower envelopes of the long-run behavior, and they are the right tools for stating asymptotic claims about bounded sequences whose oscillation does not damp out.

Definition 14. For a bounded sequence (a_n) , the *tail supremum* and *tail infimum* from index n are

$$s_n := \sup_{k \geq n} a_k, \quad i_n := \inf_{k \geq n} a_k,$$

both of which exist by the LUB property. The *limit superior* and *limit inferior* are

$$\limsup_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} s_n = \inf_{n \in \mathbb{N}} \sup_{k \geq n} a_k, \quad \liminf_{n \rightarrow \infty} a_n := \lim_{n \rightarrow \infty} i_n = \sup_{n \in \mathbb{N}} \inf_{k \geq n} a_k.$$

The two limits exist because (s_n) is monotone decreasing (the supremum over a smaller tail is at most the supremum over a larger one) and bounded below; (i_n) is monotone increasing and bounded above; both converge by monotone convergence. So lim sup and lim inf are always well-defined real numbers for any bounded sequence, even when lim is not.

The relationship to ordinary convergence is exactly what one would hope:

Proposition 15. For a bounded sequence (a_n) , $\liminf a_n \leq \limsup a_n$, with equality if and only if $\lim a_n$ exists, in which case all three are equal.

We omit the proof. The forward direction $\liminf \leq \limsup$ follows from $i_n \leq s_n$ for every n . For the equality biconditional, the “if” direction is direct from convergence; the “only if” uses the squeeze, since $i_n \leq a_n \leq s_n$ and the outer two converge to the common value.

Example 16 (A non-convergent bounded sequence). Let $a_n = (-1)^n + 1/n$ for $n \geq 1$. The even-indexed subsequence is $1 + 1/n \rightarrow 1$; the odd-indexed subsequence is $-1 + 1/n \rightarrow -1$. The sequence does not converge, but $\limsup a_n = 1$ and $\liminf a_n = -1$. The two “envelope” values are the limits of the two natural subsequences.

The example is general: $\limsup a_n$ is the largest limit of any convergent subsequence of (a_n) , and $\liminf a_n$ is the smallest. (A bounded sequence has convergent subsequences, by Bolzano–Weierstrass, and the set of subsequential limits has a maximum and a minimum, equal to lim sup

and \liminf respectively.) For a sequence that converges to L , all subsequences converge to L , so $\limsup = \liminf = L$.

The applied use of \limsup and \liminf is for sequences that do not settle but are nonetheless bounded. An approval-rating series that oscillates between, say, 40% and 55% over an indefinite stretch may not converge, but \limsup and \liminf pin down the long-run upper and lower envelopes. A sequence of empirical estimates whose noise does not vanish — because the underlying parameter itself is changing, or the sample size is bounded — has well-defined \limsup and \liminf even when \lim fails to exist. In probability theory, the Borel–Cantelli lemmas are most cleanly stated in terms of \limsup and \liminf of events; we will meet them in the probability cluster.

6 What’s next

This handout is the entry point to the analysis cluster. The two follow-ups extend the basic vocabulary in different directions:

- *Open and closed sets.* The topology of \mathbb{R} and \mathbb{R}^n — open sets, closed sets, accumulation points, closure, interior, boundary, compactness, connectedness. Convergence of sequences gives a sequential characterization of closed sets (a set is closed iff it contains all limits of sequences in it) that we will lean on; the Heine–Borel theorem (compactness in $\mathbb{R}^n =$ closed and bounded) is the higher-dimensional version of the Bolzano–Weierstrass result of §4.
- *Continuity.* The ϵ - δ definition of continuity, the sequential characterization (a function is continuous at x iff it preserves limits of sequences converging to x), the intermediate value and extreme value theorems, uniform continuity. Most of the abstract machinery applies, with continuity providing the bridge between sequences and functions.

The probability-and-measure cluster, eventually, will pick up the convergence story for random variables — where “ $X_n \rightarrow X$ ” admits multiple non-equivalent definitions (almost surely, in probability, in distribution, in L^p), each useful for different theorems. The law of large numbers and the central limit theorem are statements about which mode of convergence holds for which sequences of empirical averages.

For broader treatments at this level, see Rudin (1976) (compact and exacting — the standard graduate reference for real analysis at this level) or Abbott (2015) (gentler, with more attention to motivation; an excellent first read).

7 Exercises

Exercise 17. Prove $a_n = 1/n^2 \rightarrow 0$ from the definition. (You may use the Archimedean property of \mathbb{R} without proof.)

Exercise 18. Prove the algebra-of-limits sum rule: if $a_n \rightarrow L$ and $b_n \rightarrow M$, then $a_n + b_n \rightarrow L + M$. Where in the proof does the triangle inequality enter, and where is the choice of $\epsilon/2$ versus ϵ being made?

Exercise 19. Show that every Cauchy sequence in \mathbb{R} is bounded. (This is used inside the proof of Theorem 12; spell out the argument.)

Exercise 20 (Polling estimator). Let $\bar{x}_n := (1/n) \sum_{k=1}^n x_k$ be the running average of a deterministic sequence x_1, x_2, \dots in $[0, 1]$ (think of these as binary vote leans coded as 0 or 1, or as approval-rating reports), and suppose the underlying sequence has a Cesàro limit, meaning $\bar{x}_n \rightarrow \mu$ for some $\mu \in [0, 1]$. Show: if the x_k themselves converge to μ , then $\bar{x}_n \rightarrow \mu$ as well. (The converse can fail — \bar{x}_n can converge while x_n does not, e.g., $x_n = (-1)^n$. The probabilistic version of this exercise is the law of large numbers for i.i.d. random variables, which we will meet in the probability cluster.)

Exercise 21 (Tatonnement / price adjustment). Let $\lambda \in (0, 1)$, and define the price-adjustment sequence $p_{n+1} = (1 - \lambda)p_n + \lambda\bar{p}$, where \bar{p} is a market-clearing price and λ is the speed of adjustment. Show that $p_n \rightarrow \bar{p}$ for any starting price p_0 , and exhibit explicitly the dependence of N on ϵ in the ϵ - N definition. (Hint: $|p_n - \bar{p}| = (1 - \lambda)^n |p_0 - \bar{p}|$, a geometric sequence.)

Exercise 22 (GDP convergence). Let $\beta \in (0, 1)$ and consider $y_{n+1} = y^* + (1 - \beta)(y_n - y^*)$, the standard “ β -convergence” law of motion in growth economics: y_n is a country’s log output and y^* is the steady-state log output. Show $y_n \rightarrow y^*$. Identify the rate of convergence and state in words what a smaller β corresponds to economically.

Exercise 23. Prove that monotone convergence fails in \mathbb{Q} . That is, exhibit a monotone increasing sequence in \mathbb{Q} that is bounded above (in \mathbb{Q}), and show it does not converge to any rational number. (Hint: decimal truncations of an irrational.)

Exercise 24. Show that any nested sequence of closed bounded intervals $[c_1, d_1] \supseteq [c_2, d_2] \supseteq \dots$ in \mathbb{R} has nonempty intersection. (This is the *nested intervals theorem*, used inside the proof of Bolzano–Weierstrass; supply the argument from the LUB property directly.)

Exercise 25 (Oscillating approval rating). Let a_n be the approval rating of an executive in month n , and suppose $a_{2k} = 0.55 + 1/(k + 1)$ and $a_{2k+1} = 0.40 - 1/(k + 1)$ for $k \geq 0$. Compute $\limsup_n a_n$ and $\liminf_n a_n$. Does the sequence converge? In a sentence, what is the substantive content of the gap $\limsup - \liminf$ for someone trying to summarize the executive’s standing?

Exercise 26. Show that for any bounded sequence (a_n) , $\limsup a_n$ equals the supremum of the set of subsequential limits — i.e., the supremum over all L such that some subsequence $a_{n_k} \rightarrow L$. (You may use Proposition 7 and the definition of \limsup as $\inf_n \sup_{k \geq n} a_k$.)

References

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