# Propositional logic

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Last revised: 5 May 2026

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## 1 Motivation

Political and economic theorists make claims of the form "if such-and-such, then so-and-so." To say a conclusion *follows from* a set of premises is to make a logical claim, separable from any empirical content. Propositional logic is the simplest formal system in which this notion of "follows from" can be made precise. Everything that comes later in these notes presupposes some facility with it.

This handout introduces the language of propositional logic, its semantics, and the central notions of tautology, satisfiability, equivalence, and logical consequence. The work is mostly definitional — there are not many surprises along the way — but the definitions have to be airtight, because everything that follows in these notes sits on top of them. Proof systems and first-order extensions are taken up in separate handouts.

## 2 Language

Before we can ask what a sentence *means*, we have to settle what counts as a sentence at all. The temptation is to gloss over this — surely we know what an "if-then" statement looks like — but glossing over it is what produces the worst confusions later. So we'll be explicit: alphabet, connectives, and the rules for combining them. Pure bookkeeping, but the inductive structure laid down here is what makes everything later tractable.

**Definition 1.** The *language of propositional logic* consists of:

- a countable set[1] $\mathcal{P} = \{p_1, p_2, \dots\}$ of *atomic propositions* (also called *propositional variables*);
- the *connectives* $\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$;
- parentheses ( and ).

**Definition 2.** The set of *well-formed formulas* (*wffs*) is the smallest set $\Phi$ such that:

- every atomic proposition is in $\Phi$;
- if $\varphi \in \Phi$, then $(\neg\varphi) \in \Phi$;
- if $\varphi, \psi \in \Phi$, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \to \psi)$, and $(\varphi \leftrightarrow \psi)$ are all in $\Phi$.

We use $\varphi, \psi, \chi, \dots$ as variables ranging over wffs.

We omit parentheses when no ambiguity results, using standard precedence: $\neg$ binds tightest, then $\wedge$, then $\vee$, then $\to$, then $\leftrightarrow$. Thus $p \wedge q \to r$ abbreviates $((p \wedge q) \to r)$, not $(p \wedge (q \to r))$.

[1] Set-theoretic vocabulary ("set," "function," etc.) is used informally throughout, prior to the handout on naive set theory. See the project preface for discussion.

## 3 Semantics

Syntax tells us which strings count as formulas. Semantics tells us when a formula is true. For propositional logic the answer to "true under what?" is a *valuation* — a function from atomic propositions to truth values — and the rules for propagating from atoms to complex formulas are completely mechanical. That mechanical character is the point: it is why propositional logic is a useful starting place. There is no judgment call about what a connective means; the truth tables fix it.

A *truth value* is one of $\{T, F\}$. A *valuation* (also *model*, *interpretation*, or *truth assignment*) is a function $v : \mathcal{P} \to \{T, F\}$ that assigns a truth value to each atomic proposition.

**Definition 3.** A valuation $v$ extends to a function $\hat{v} : \Phi \to \{T, F\}$ on all wffs, defined recursively:[2]

$$\begin{aligned}
\hat{v}(p) &= v(p) \quad \text{for } p \in \mathcal{P} \\
\hat{v}(\neg\varphi) = T &\iff \hat{v}(\varphi) = F \\
\hat{v}(\varphi \wedge \psi) = T &\iff \hat{v}(\varphi) = T \text{ and } \hat{v}(\psi) = T \\
\hat{v}(\varphi \vee \psi) = T &\iff \hat{v}(\varphi) = T \text{ or } \hat{v}(\psi) = T \\
\hat{v}(\varphi \to \psi) = T &\iff \hat{v}(\varphi) = F \text{ or } \hat{v}(\psi) = T \\
\hat{v}(\varphi \leftrightarrow \psi) = T &\iff \hat{v}(\varphi) = \hat{v}(\psi)
\end{aligned}$$

By a standard abuse of notation, we write $v(\varphi)$ for $\hat{v}(\varphi)$.

The connectives are most easily summarized by their *truth tables*:

| *given data* | | *operators defined* | | | | |
|---|---|---|---|---|---|---|
| $\varphi$ | $\psi$ | $\neg\varphi$ | $\varphi \wedge \psi$ | $\varphi \vee \psi$ | $\varphi \to \psi$ | $\varphi \leftrightarrow \psi$ |
| $T$ | $T$ | $F$ | $T$ | $T$ | $T$ | $T$ |
| $T$ | $F$ | $F$ | $F$ | $T$ | $F$ | $F$ |
| $F$ | $T$ | $T$ | $F$ | $T$ | $T$ | $F$ |
| $F$ | $F$ | $T$ | $F$ | $F$ | $T$ | $T$ |

*Remark.* The material conditional $\to$ deserves special attention. In ordinary English, "if $\varphi$ then $\psi$" often carries a connotation of relevance or causation: we mean the antecedent has something to do with the consequent. The material conditional has no such constraint. $\varphi \to \psi$ is true whenever $\varphi$ is false, regardless of $\psi$. "If the moon is made of cheese, then every voter is rational" is a true sentence of propositional logic. The vacuous case is occasionally a source of confusion in informal arguments.

[2] This style of recursive definition — specifying truth-in-a-valuation by induction on the structure of formulas — is a propositional-logic version of what Tarski (1933) called a *definition of truth*. Tarski's enduring move was to insist that any such definition has to be given in a metalanguage strictly stronger than the object language being defined; otherwise paradox lurks. (The canonical demonstration is the liar sentence: "this sentence is false.") For propositional logic the danger is mostly theoretical — the language is too weak to refer to its own truth predicate. But the same style of definition extended to first-order arithmetic runs straight into Tarski's undefinability theorem: no sufficiently expressive theory of arithmetic can define its own truth predicate. We will not need that result here; the point is that the recursion below is the harmless propositional ancestor of something much deeper.

## 4 Tautology, contradiction, satisfiability

With truth-in-a-valuation in hand, three sorting questions present themselves immediately. Is this formula true on every valuation? On none? On at least one? These three properties — being a tautology, a contradiction, or satisfiable — partition the space of formulas, and they do most of the heavy lifting in formal arguments. Tautologies are the logical truths; contradictions are the logical falsehoods; satisfiability is the workhorse for talking about whether a set of claims is jointly coherent.

**Definition 4.** A wff $\varphi$ is:

- a *tautology* if $v(\varphi) = T$ for every valuation $v$;
- a *contradiction* if $v(\varphi) = F$ for every valuation $v$;
- *satisfiable* if $v(\varphi) = T$ for some valuation $v$;
- *contingent* if it is neither a tautology nor a contradiction.

**Example 5.** $p \vee \neg p$ is a tautology (the law of excluded middle); $p \wedge \neg p$ is a contradiction; $p \to (q \to p)$ is a tautology; $p \to q$ is contingent.

**Example 6** (Bicameralism as a propositional formula)**.** Let $H$ stand for "the bill passes the House," $S$ for "the bill passes the Senate," and $L$ for "the bill becomes law." Setting aside the veto, a stylized statement of bicameral passage is $L \leftrightarrow (H \wedge S)$. This formula is contingent: there are valuations making it true (e.g., $H, S, L$ all $T$) and valuations making it false (e.g., $H, S$ true but $L$ false—corresponding to a veto that the formula does not yet capture). Refining the formula to handle the veto is left as an exercise.

**Exercise 7.** Extend the bicameralism example: introduce $V$ for "the president vetoes the bill" and $O$ for "Congress overrides the veto by a two-thirds majority in each chamber." Write a propositional formula for $L$ that captures the U.S. constitutional rule, and verify that it is contingent.

## 5 Logical equivalence

"These two formulas say the same thing" is the kind of judgment people make casually all the time, and one that has to be sharpened before it can carry weight in a proof. The definition is what you would expect: two formulas are equivalent if they have the same truth value on every valuation. The payoff is a small list of standard equivalences — de Morgan, contraposition, and friends — that lets you rewrite formulas without changing what they claim. Most of what propositional reasoning actually feels like is recognizing these equivalences in the wild.

**Definition 8.** Two wffs $\varphi$ and $\psi$ are *logically equivalent*, written $\varphi \equiv \psi$, if $v(\varphi) = v(\psi)$ for every valuation $v$. Equivalently, $\varphi \leftrightarrow \psi$ is a tautology.

**Proposition 9** (Standard equivalences)**.** *For all wffs $\varphi$, $\psi$, $\chi$:*

1. *Double negation:* $\neg\neg\varphi \equiv \varphi$.
2. *De Morgan's laws:* $\neg(\varphi \wedge \psi) \equiv \neg\varphi \vee \neg\psi$ *and* $\neg(\varphi \vee \psi) \equiv \neg\varphi \wedge \neg\psi$.

3. *Contraposition:* $(\varphi \to \psi) \equiv (\neg\psi \to \neg\varphi)$.

4. *Implication as disjunction:* $(\varphi \to \psi) \equiv (\neg\varphi \vee \psi)$.

5. *Distribution:* $\varphi \wedge (\psi \vee \chi) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$, *and the same with* $\wedge$ *and* $\vee$ *exchanged.*

*Proof.* Each is verified by truth table: the columns under the left- and right-hand sides agree on every row. □

**Example 10** (Contraposition in political argument)**.** The claim "if a regime is democratic, it does not initiate war against another democratic regime" is contrapositively equivalent to "if a regime initiates war against another democratic regime, it is not itself democratic." The two are interchangeable as logical statements, though they may suggest different empirical strategies for testing.

## 6 Logical consequence

This is what we came for. The whole point of syntax-then-semantics was to give a precise account of what it means for a conclusion to *follow from* a set of premises. The definition is the natural one once valuations are in hand: the conclusion follows if every valuation that makes all the premises true also makes the conclusion true. From there we can talk about which inference patterns are valid in general — modus ponens, modus tollens, and the elementary toolkit — and then about specific applications.

**Definition 11.** A wff $\psi$ is a *logical consequence* of a set $\Gamma$ of wffs, written $\Gamma \models \psi$,[3] if every valuation that makes every formula in $\Gamma$ true also makes $\psi$ true.

When $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ is finite we write $\varphi_1, \ldots, \varphi_n \models \psi$. The notation $\models \psi$ (no premises) means $\psi$ is a tautology.

It is convenient to have compact notation for iterated conjunction and disjunction: we write $\bigwedge_{i=1}^{n} \varphi_i$ for $(\varphi_1 \wedge \cdots \wedge \varphi_n)$, and similarly $\bigvee_{i=1}^{n} \varphi_i$ for the iterated disjunction. The meaning is the obvious one — the same connective applied across the indexed family.

**Proposition 12.** $\varphi_1, \ldots, \varphi_n \models \psi$ *if and only if* $\bigwedge_{i=1}^{n} \varphi_i \to \psi$ *is a tautology.*

*Proof.* Both sides assert the same condition on valuations: every $v$ satisfying all the $\varphi_i$ also satisfies $\psi$. □

**Example 13** (Modus ponens)**.** $\varphi, \varphi \to \psi \models \psi$. Any valuation making both $\varphi$ and $\varphi \to \psi$ true must, by the truth table for $\to$, make $\psi$ true.

[3]What we have defined is *semantic* implication: the relation $\models$ is about valuations and truth. There is a parallel *syntactic* notion, written $\Gamma \vdash \psi$ (single turnstile), that holds when $\psi$ can be *derived* from $\Gamma$ using a fixed set of formal inference rules — $\vee$-elimination, reductio ad absurdum, double-negation elimination, and the rest. Different proof systems — natural deduction, sequent calculus, Hilbert-style axiomatic presentations — package the rules differently. The remarkable fact, established by soundness and completeness theorems, is that the two notions coincide: $\Gamma \vdash \psi$ if and only if $\Gamma \models \psi$. For the working political theorist or formal modeler, however, the syntactic apparatus is mostly illuminating rather than load-bearing: once you can reason about valuations, the explicit rule-by-rule derivations do not add much to the toolkit. They are worth meeting once — see the separate handout on proof systems — but most applications run on the semantic side.

**Example 14** (Modus tollens)**.** $\neg\psi, \varphi \to \psi \models \neg\varphi$ — the contrapositive form of modus ponens. By the proposition above, this is equivalent to saying that $((\varphi \to \psi) \wedge \neg\psi) \to \neg\varphi$ is a tautology, which we can verify directly:

| *given* | | *intermediates* | | | | *full formula* |
|---|---|---|---|---|---|---|
| $\varphi$ | $\psi$ | $\varphi \to \psi$ | $\neg\psi$ | $(\varphi \to \psi) \wedge \neg\psi$ | $\neg\varphi$ | $((\varphi \to \psi) \wedge \neg\psi) \to \neg\varphi$ |
| $T$ | $T$ | $T$ | $F$ | $F$ | $F$ | $T$ |
| $T$ | $F$ | $F$ | $T$ | $F$ | $F$ | $T$ |
| $F$ | $T$ | $T$ | $F$ | $F$ | $T$ | $T$ |
| $F$ | $F$ | $T$ | $T$ | $T$ | $T$ | $T$ |

The final column is $T$ on every row, so the formula is a tautology.

**Example 15** (Democratic peace as a logical consequence)**.** Let $D$ stand for "the regime is democratic" and $W$ for "the regime initiates war against another democracy." Suppose we accept the (much-debated) democratic peace claim $D \to \neg W$. Then $W \models \neg D$: observing that a regime initiated war against another democracy logically forces the conclusion that it was not itself democratic. Whether to accept the premise is an empirical question; whether the conclusion follows from the premise is a logical one. Confusing the two is a common mistake.

## 7 What's next

Two strands extend this handout:

- *Proof systems.* Logical consequence is a semantic notion: it ranges over valuations. *Proof* is the syntactic counterpart—a finite combinatorial procedure that, when correctly applied, generates exactly the consequences. The soundness and completeness theorems link the two. (Separate handout.)
- *First-order logic.* Propositional logic treats whole sentences as atomic. To say things like "every voter has a preference" or "there exists a Condorcet winner," we need predicates, variables, and quantifiers. (Separate handout.)

For a fuller mathematical treatment, see Enderton (2001).

## 8 Exercises

**Exercise 16.** Verify by truth table that hypothetical syllogism — $((\varphi \to \psi) \wedge (\psi \to \chi)) \to (\varphi \to \chi)$ — is a tautology.

**Exercise 17.** For each formula below, decide whether it is a tautology, a contradiction, or contingent. Justify briefly (a truth table is fine).

1. $(p \wedge q) \to (p \vee q)$
2. $(p \to q) \leftrightarrow (\neg p \vee q)$

3. $(p \to q) \wedge (p \wedge \neg q)$
4. $(p \vee q) \to (p \wedge q)$

**Exercise 18.** Verify by truth table that $(p \to q) \wedge (p \to r) \equiv p \to (q \wedge r)$.

**Exercise 19.** Article V of the U.S. Constitution sets out (in stylized form) the rule for amending the Constitution: an amendment is enacted if and only if it is *proposed* and *ratified*. It may be proposed either by a two-thirds vote in each chamber of Congress or by a convention called by two-thirds of state legislatures; it must be ratified by three-fourths of state legislatures. Let $A$ stand for "the amendment is enacted," $C$ for "two-thirds of each chamber of Congress proposes it," $N$ for "two-thirds of state legislatures call a convention proposing it," and $R$ for "three-fourths of state legislatures ratify it." Write a propositional formula expressing the rule, and verify that it is contingent.

**Exercise 20.** Prove from the definition of $\models$ that $p \to q,\ q \to r \models p \to r$. Argue about valuations directly — do not appeal to any proof system.

**Exercise 21.** Let $p_1, \ldots, p_n$ stand for "party $i$ supports the proposed reform," for $i = 1, \ldots, n$.

1. Translate "at least one party supports the reform" using $\bigvee$.
2. Translate "every party supports the reform" using $\bigwedge$.
3. Translate "exactly one party supports the reform." (You will need both $\bigvee$ and $\bigwedge$.)

**Exercise 22.** Consider the argument:

> If the bill is funded, it passes. The bill is not funded. Therefore, the bill does not pass.

Translate the premises and conclusion to propositional logic and determine whether the conclusion is a logical consequence of the premises. If it is not, exhibit a valuation on which the premises are true but the conclusion is false. (The fallacy has a name; you may have encountered it.)

**Exercise 23.** Show that logical consequence is *monotonic*: if $\Gamma \models \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \models \psi$. Argue directly from the definition.

**Exercise 24.** Show that disjunctive syllogism is valid: $\varphi \vee \psi,\ \neg\varphi \models \psi$.

**Exercise 25.** In a stylized parliamentary system, the government falls if and only if it loses a vote of no confidence or the prime minister resigns. Let $G$ stand for "the government falls," $V$ for "the government loses a vote of no confidence," and $R$ for "the prime minister resigns."

1. Write the rule as a propositional formula.
2. Suppose $\neg G$. What can be concluded about $V$ and $R$? State the conclusion as a single formula and verify, by reasoning about valuations, that it is a logical consequence of the rule together with $\neg G$.
3. Now drop one direction of the rule: assume only that the government falls *if* it loses a no-confidence vote or the PM resigns (but possibly for other reasons too). Does the conclusion in (2) still follow? Why or why not?

## References

Enderton, Herbert B. (2001). *A Mathematical Introduction to Logic*. 2nd ed. San Diego: Academic Press.