Order theory

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1 Motivation

Political and economic theorists are constantly comparing things. One alternative is at least as good as another. One coalition contains another. One outcome dominates another. One strategy weakly dominates another in a game. Each of these is a binary relation with structure — typically reflexivity, transitivity, and sometimes antisymmetry or comparability — and the family of relations with this kind of structure is the subject of order theory.

The vocabulary shows up in two main places in formal political economy. First, in *preferences*: a preference relation is a particular kind of order, and the formal theory of choice rides on its order-theoretic properties — which axioms hold, which don't, which combinations admit a real-valued utility representation. Second, in *dominance*: Pareto dominance, stochastic dominance, game-theoretic strict and weak dominance — these are partial orders, and the conditions for an outcome to be optimal (efficient, undominated, unanimously preferred) are stated order-theoretically.

This handout develops the preorder / partial order / total order hierarchy, the strict-and-symmetric decomposition that goes with each, the visual apparatus of Hasse diagrams, bounds and suprema (with the maximum-versus-maximal distinction that matters whenever the order is not total), lattices in brief, and order-preserving functions — with utility representation as the headline application.

2 Preorders, partial orders, total orders

The relevant axioms are few — reflexive, antisymmetric, transitive, comparable — and the work is in noticing which combinations matter and what each is good for. We build upward from the weakest order structure to the strongest. Each stage adds an axiom and rules out a kind of pathology that the previous stage allowed.

Definition 1. A binary relation $\preceq$ on a set A is a:

- *preorder* if it is reflexive and transitive;
- *partial order* if it is reflexive, antisymmetric, and transitive;
- *total order* (or *linear order*) if it is a partial order and additionally satisfies *comparability*: for every $a, b \in A$, either $a \preceq b$ or $b \preceq a$.

We call the pair $(A, \preceq)$ a *preordered set*, *partially ordered set* (*poset*, for short), or *totally ordered set* according to which axioms $\preceq$ satisfies.

Example 2 (Subsets)**.** On $\mathcal{P}(A)$ for any set A, the relation $\subseteq$ is a partial order: reflexive ($S \subseteq S$), antisymmetric (mutual inclusion implies equality, by extensionality), and transitive. It is not a total order as soon as $|A| \geq 2$: $\{a\}$ and $\{b\}$ are incomparable.

Example 3 (Divisibility)**.** On the positive integers, define $a \mid b$ to mean "a divides b," i.e., $b = ak$ for some positive integer k. This is a partial order: reflexive ($a \mid a$ via $k = 1$), antisymmetric (mutual divisibility on positive integers forces equality), and transitive. It is not total: 2 and 3 are incomparable.

Example 4 (The reals)**.** $(\mathbb{R}, \leq)$ is a total order, as are $(\mathbb{Z}, \leq)$ and $(\mathbb{Q}, \leq)$.

Example 5 (Pareto dominance)**.** For $x, y \in \mathbb{R}^n$, define $x \preceq^P y$ to mean $x_i \leq y_i$ for every $i = 1, \ldots, n$ (*weak Pareto dominance*). This is a partial order on $\mathbb{R}^n$, and is total only if $n = 1$. The associated strict relation $\prec^P$ ($x_i \leq y_i$ for all i, with strict inequality at some j) is what political economists usually call *Pareto dominance* when the x_i are interpreted as the utilities of n agents.

A preorder that is *not* a partial order is one in which two distinct elements can each be at least as "high" as the other. Antisymmetry rules this out. The most important place where antisymmetry is the wrong assumption is in preferences:

Example 6 (Preferences)**.** A preference relation $\succsim$ on a set X of alternatives is conventionally taken to be a *complete preorder*: reflexive, transitive, and additionally satisfying *completeness* — $x \succsim y$ or $y \succsim x$ for every $x, y \in X$. Antisymmetry would force any two indifferent alternatives to be equal, which is the wrong thing: distinct alternatives can be perceived as equally good without becoming the same alternative. So preferences are preorders, not partial orders, and indifference is a meaningful relation rather than identity.[1]

Every preorder decomposes into a strict part and a symmetric part, and these capture different content.

Definition 7. For a preorder $\preceq$ on A, the *strict part* $\prec$ is defined by

$$a \prec b \iff a \preceq b \text{ and not } b \preceq a,$$

and the *symmetric part* $\sim$ is defined by

$$a \sim b \iff a \preceq b \text{ and } b \preceq a.$$

Proposition 8. *Let $\preceq$ be a preorder on A. Then:*

1. *The strict part $\prec$ is irreflexive and transitive.*
2. *The symmetric part $\sim$ is an equivalence relation.*
3. *For any $a, b \in A$, exactly one of the following holds: $a \prec b$, $b \prec a$, $a \sim b$, or a and b are* incomparable *(neither $a \preceq b$ nor $b \preceq a$).*

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[1] The label "complete preorder" is sometimes also called "weak order" or "total preorder"; all three names point at the same structure: reflexive + transitive + complete. The completeness axiom is informally the assumption that the agent always has a view — given any two alternatives, the agent can compare them. Several settings make this awkward enough that completeness is dropped: choice under deep uncertainty (the agent has not formed a view about every conceivable contingency), multi-criteria decision making (alternatives are ranked on several dimensions and there is no commitment to aggregating them), and unanimity-style social rankings (the social planner's order is the intersection of individual orders, which is partial whenever individuals disagree). When completeness is dropped, the preference relation becomes a partial order rather than a complete preorder, and the basic existence theorems for choice (e.g., utility representation, see §6) need significant modification or fail outright. The convention in mainstream economic theory is to assume completeness; the convention in social choice and decision theory under ambiguity is to take dropping it seriously. Both are doing order theory; they are just doing it on different objects.

Proof. **(1)** **Irreflexivity:** if $a \prec a$, then $a \preceq a$ and not $a \preceq a$ — contradiction. **Transitivity:** if $a \prec b$ and $b \prec c$, then $a \preceq b$ and $b \preceq c$, so $a \preceq c$ by transitivity of $\preceq$. We also need $\neg(c \preceq a)$: if $c \preceq a$, then $c \preceq b$ by transitivity, contradicting $b \prec c$. So $a \prec c$.

(2) Reflexive ($a \preceq a$ gives $a \sim a$); symmetric (immediate from the definition); transitive (if $a \sim b$ and $b \sim c$, transitivity of $\preceq$ in both directions gives $a \sim c$).

(3) Each pair a, b falls into exactly one of the four cases of the truth table on whether $a \preceq b$ and $b \preceq a$ hold. □

For preferences in particular, the strict part is what is usually written $\succ$ (*strict preference*), and the symmetric part is $\sim$ (*indifference*). Proposition 8 tells us indifference is an equivalence relation: indifference classes partition the alternative space, and $\succ$ is a strict order *on* indifference classes. When $\succsim$ is complete, the "incomparable" case never arises, so any two alternatives are either strictly ordered or indifferent.

Total orders, finally, are partial orders that admit no incomparability at all.

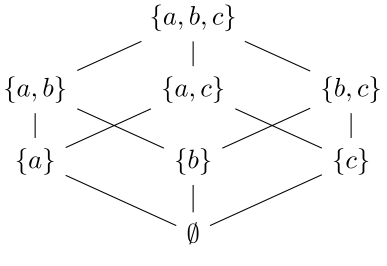
Example 9 (Lexicographic order)**.** On $\mathbb{R}^n$, define $x \preceq^L y$ if either $x = y$, or there exists some i such that $x_j = y_j$ for all $j < i$ and $x_i < y_i$. This is a total order: every distinct pair is strictly comparable in exactly one direction. Lexicographic preferences — prefer x to y if you prefer the first component, ties broken by the second — are a classic example, and a usefully pathological one for utility representation (see §6).

3 Hasse diagrams

A finite (or otherwise sufficiently small) partial order can be drawn on the page in a way that makes its structure immediately visible. Elements are placed so that "higher in the order" becomes "higher on the page," and lines are drawn between immediate predecessors and successors. The resulting picture is a *Hasse diagram*, and it is the standard expository tool for thinking about partial orders.

Definition 10. In a poset $(A, \preceq)$, b *covers* a (written $a \lessdot b$) if $a \prec b$ and there is no c with $a \prec c \prec b$. The *Hasse diagram* of $(A, \preceq)$ is the picture obtained by placing the elements on the page so that a is below b whenever $a \prec b$, and drawing a line from a to b for each covering pair $a \lessdot b$. By convention the lines have no arrowheads — the order is read off the page, with $a \preceq b$ holding iff there is an upward path from a to b.

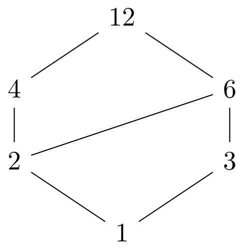
The Hasse diagram of $(\mathcal{P}(\{a, b, c\}), \subseteq)$:



The eight subsets sit at four levels (corresponding to subset size), with covering edges connecting each subset to those obtained by adding one element. The diagram of $(\mathcal{P}(A), \subseteq)$ for any finite A

is the $|A|$-dimensional Boolean cube; for $|A| = 3$ that's the picture above, and the lines crossing between the second and third levels are unavoidable in any 2-D drawing.

The Hasse diagram of divisibility on $\{1, 2, 3, 4, 6, 12\}$:



This diagram makes the partial-but-not-total character of divisibility visible at a glance: 4 and 6 have no edge between them, and indeed $4 \nmid 6$ and $6 \nmid 4$. A total order, by contrast, has a Hasse diagram that is just a line, and there is little to learn from drawing it.

4 Bounds, maxima, suprema

In a poset, "the highest element of a subset" splits into two notions, and "the smallest thing above a subset" splits into another two. For a totally ordered set the four notions collapse to one. For a genuinely partial order they don't, and the difference is what the language of bounds and suprema is for.

Definition 11. Let $(A, \preceq)$ be a poset and $S \subseteq A$. An *upper bound* of S in A is an element $u \in A$ with $s \preceq u$ for every $s \in S$. The *supremum* (or *least upper bound*, $\sup S$) is an upper bound of S that is $\preceq$ every other upper bound. *Lower bounds* and the *infimum* ($\inf S$) are defined dually.

The supremum, when it exists, is unique by antisymmetry. Existence is not guaranteed: a subset can have no upper bound at all (so no supremum), or have several upper bounds but no least one.

Definition 12. Let $(A, \preceq)$ be a poset and $S \subseteq A$. An element $a \in S$ is:

- the *maximum* of S if $b \preceq a$ for every $b \in S$;
- *maximal in* S if no $b \in S$ has $a \prec b$.

Minimum and *minimal* are defined dually.

Maximum and maximal coincide in a total order but diverge in a genuinely partial one. A maximum is unbeaten by every other element of S, including the ones it is incomparable with. A maximal element is unbeaten by anyone in S *who is comparable to it* — if there are elements of S incomparable to a maximal a, the maximal status is undisturbed. So maximal elements always exist in finite posets,[2] but maxima may not exist when the order is genuinely partial.

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[2] In an infinite poset, the existence of maximal elements is not automatic: $(\mathbb{Z}, \leq)$ has none. The standard sufficient condition is *Zorn's lemma*: a poset in which every chain (totally ordered subset) has an upper bound contains a maximal element. Zorn's lemma is logically equivalent to the axiom of choice; we will not need it actively here, but it is the workhorse for non-constructive existence proofs in algebra and analysis (every vector space has a basis; every nonzero ring has a maximal ideal).

Example 13 (Pareto-optimal alternatives)**.** Let $X \subseteq \mathbb{R}^n$ be a set of feasible alternatives, with x_i interpreted as the utility of agent i, and order X by Pareto dominance: $x \preceq^P y$ iff $x_i \leq y_i$ for every i. A *Pareto-optimal* (*Pareto-efficient*) alternative is a maximal element of $(X, \preceq^P)$ — one not strictly Pareto-dominated by any other feasible alternative. There need not be a maximum: in a generic feasible set, no single alternative is at least as good for every agent as every other alternative. The set of Pareto-optimal points is the *Pareto frontier* of X.

Example 14 (Suprema in $\mathbb{R}$)**.** In $(\mathbb{R}, \leq)$, every nonempty subset that is bounded above has a supremum. This is the *least upper bound property*, and it is the structural fact that distinguishes $\mathbb{R}$ from $\mathbb{Q}$: the rationals do not have this property (consider $\{q \in \mathbb{Q} : q^2 \leq 2\}$, which has rational upper bounds but no least rational upper bound). The least upper bound property is the order-theoretic backbone of real analysis.

5 Lattices

Lattices show up wherever a partial order comes paired with a natural "meet" and "join" that always exist: the smallest coalition containing two given coalitions, the intersection of two upper-contour sets, the highest common policy that two factions both prefer. Each is a meet or a join in some underlying lattice, and recognizing the structure converts an order-theoretic question into an algebraic one. The standard examples — power sets under inclusion, real numbers under $\leq$, the divisibility order on positive integers — are all lattices, and we will use them as the running illustrations.

Definition 15. A poset $(A, \preceq)$ is a *lattice* if every pair $a, b \in A$ has a supremum $a \vee b := \sup\{a, b\}$ (the *join*) and an infimum $a \wedge b := \inf\{a, b\}$ (the *meet*). The lattice is *complete* if every subset $S \subseteq A$ (not just every pair) has a supremum and an infimum.

Example 16. $(\mathcal{P}(A), \subseteq)$ is a complete lattice: $\bigcup_i S_i$ is the supremum and $\bigcap_i S_i$ the infimum of any family of subsets.

Example 17. $(\mathbb{R}, \leq)$ is a lattice with $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$, but it is not complete: an unbounded subset has no supremum. The least-upper-bound property amounts to saying every nonempty subset that is bounded above has a supremum — a partial completeness, sufficient for analysis.

Example 18. On the positive integers ordered by divisibility, $a \vee b = \operatorname{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$. (Verify against the Hasse diagram in §3.)

Lattices show up in political economy mostly via fixed-point arguments: Tarski's theorem says every order-preserving function on a complete lattice has a fixed point, which is the workhorse for existence proofs in models with strategic complementarity (supermodular games). The depth here is real, but it lives in a separate handout; the definitions above are what is needed to read the statement.

6 Order-preserving functions

Functions between ordered sets fall into two flavors: those that respect the order and those that don't. The former are the relevant kind for almost every application: a utility function should be order-preserving with respect to the underlying preference; a price function should be monotone in quantity demanded; a function translating between two ordered indexes should respect the structure on both sides.

Definition 19. Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets. A function $f : A \to B$ is:

- *order-preserving* (or *monotone*, or *isotone*) if $a \preceq_A a'$ implies $f(a) \preceq_B f(a')$;
- *strictly order-preserving* if $a \prec_A a'$ implies $f(a) \prec_B f(a')$;
- an *order isomorphism* if it is a bijection and both f and f^{-1} are order-preserving.

The headline application for political economy is utility representation. A utility function is exactly an order-preserving function from the alternative space (with its preference order) to $\mathbb{R}$ (with the usual order).

Definition 20. Let $\succsim$ be a preference relation on a set X. A function $u : X \to \mathbb{R}$ *represents* $\succsim$ if for every $x, y \in X$,

$$x \succsim y \iff u(x) \geq u(y).$$

When does a representation exist?[3]

The quick answer: for finite X, always (rank the indifference classes and assign them ascending integers). For infinite X, additional structure is needed — in particular, a kind of separability — and there are familiar preferences with no real-valued representation at all.

Example 21 (Utility representation, finite case)**.** Let $X = \{x_1, x_2, x_3\}$ with $x_1 \succ x_2 \sim x_3$. Then $u(x_1) = 2$, $u(x_2) = u(x_3) = 1$ represents $\succsim$, and any other assignment respecting the order does too: representations are unique only up to a strictly increasing transformation, and the convention is to treat them as defined "up to monotone transformation" rather than as specific real numbers.

Example 22 (Lexicographic preferences are non-representable)**.** Let $X = [0, 1] \times [0, 1]$ with the lexicographic order from §2. There is no real-valued utility function representing $\preceq^L$. The intuition: the strict ordering on the first coordinate alone has uncountably many strict-preference jumps (one for every value of the first coordinate), and these cannot all be embedded in $\mathbb{R}$ without crushing some of them. The footnote above gives the formal version of the argument.

7 What's next

Two strands extend this handout:

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[3] The classical result, due to Debreu (1954), is that a complete preorder on a topological space X admits a continuous utility representation if and only if it is *continuous* (the upper- and lower-contour sets are closed in X) and X has a countable order-dense subset — a countable set $D \subseteq X$ such that, for every $x \prec y$, there is $d \in D$ with $x \succsim d \succsim y$. Without continuity, the same separability condition (a countable order-dense subset) is the content of an earlier theorem of Cantor: any complete preorder with a countable order-dense subset admits a real-valued representation. The lexicographic order on $[0, 1]^2$ fails the order-density condition: between any $(a, b) \prec^L (a, b')$ in the same first-coordinate fiber, no point with a different first coordinate sits between them, so a countable set cannot be dense across all the fibers. As a result, lexicographic preferences admit *no* real-valued utility representation, continuous or not. The moral for applied work: writing $u(x)$ in place of $\succsim$ is a real assumption, not a notational change. You are silently invoking conditions on $\succsim$ (completeness, transitivity, separability, possibly continuity) that make the representation possible. When a model fails a condition, the utility function disappears with it.

- *Real analysis.* The least upper bound property of $\mathbb{R}$ is the order-theoretic backbone of analysis, and the material that builds on it spans a cluster: *sequences and limits* (convergence, Cauchy sequences, monotone convergence), *open and closed sets* (the topology of $\mathbb{R}$ and $\mathbb{R}^n$, including compactness), and *continuity* (ϵ-δ, intermediate value, extreme value, uniform continuity). The construction of $\mathbb{R}$ from $\mathbb{Q}$ and the LUB property itself live as a section and an extended footnote inside the sequences handout rather than as separate handouts. (Separate cluster, eventually.)
- *Decision theory.* The basic theory of choice spans a cluster of handouts. *Choice under certainty* is the natural extension of the present handout: it puts preference, utility, and choice (in the sense of selection from a menu) into a single picture, with revealed preference as the bridge between observable choice behavior and the underlying preference relation — the WARP / SARP / GARP results live here, because they *are* the linkages. *Choice under risk* (probabilities given — the von Neumann–Morgenstern axioms, expected utility, the Allais paradox, prospect theory and behavioral departures within the EU framework) and *choice under ambiguity* (probabilities unknown — Savage's subjective expected utility, Ellsberg's paradox, Knightian uncertainty, multiple-prior models) extend the framework once uncertainty enters. (Separate cluster of handouts, eventually.)

For broader treatments of order theory itself, see Halmos (1960) for the elementary set-theoretic side and Davey and Priestley (2002) for a full development with lattice-theoretic depth.

8 Exercises

Exercise 23. On $\mathbb{Z}$, define $a \mathrel{R} b$ iff $a^2 \leq b^2$. Show that R is a preorder but not a partial order. Identify the symmetric part and describe the equivalence classes.

Exercise 24. Show that for a strict preorder $\prec$ (irreflexive and transitive), $\prec$ is asymmetric: $a \prec b$ implies not $b \prec a$. Conclude that asymmetry follows from irreflexivity plus transitivity, and need not be assumed separately.

Exercise 25. Let $(A, \preceq_A)$ and $(B, \preceq_B)$ be posets. Define a relation on $A \times B$ by $(a, b) \preceq (a', b')$ iff $a \preceq_A a'$ and $b \preceq_B b'$ (the *product order*). Verify that this is a partial order. Show that it is total if and only if at most one of A or B has more than one element. (Pareto dominance on $\mathbb{R}^n$ is the special case where each $\preceq_i$ is the standard order on $\mathbb{R}$; multi-criteria evaluations of policies, where each criterion is independently ranked, instantiate the same construction.)

Exercise 26. Draw the Hasse diagram of the coalition lattice for a four-member legislature: $(\mathcal{P}(\{1, 2, 3, 4\}), \subseteq)$. (Hint: 16 vertices in 5 levels.) Under simple majority rule (winning coalition = strictly more than half), identify the *minimal winning coalitions* — the coalitions that win but contain no smaller winning sub-coalition.

Exercise 27. Give an example of a finite poset with multiple maximal elements but no maximum — a setting where there are several Pareto-efficient outcomes and no single outcome dominates all others. Give an example of an infinite poset with no maximal element at all (think of an unbounded resource setting where "higher" is always available).

Exercise 28. Let $X = \{(1, 0), (0, 1), (1, 1), (2, 0), (0, 2)\} \subseteq \mathbb{R}^2$, ordered by componentwise $\leq$. List all maximal elements (the Pareto frontier of X). Is there a maximum? What are the suprema and infima of X in $(\mathbb{R}^2, \preceq^P)$, if they exist?

Exercise 29. Verify that on the positive integers ordered by divisibility, $a \vee b = \mathrm{lcm}(a, b)$ and $a \wedge b = \gcd(a, b)$.

Exercise 30. Show that the composition of two order-preserving functions is order-preserving. Then exhibit an order-preserving bijection that is not an order isomorphism. (Hint: take $A = \{0, 1\}$ with the discrete order — no comparisons except reflexive ones — and $B = \{0, 1\}$ with $0 < 1$. The identity is order-preserving from A to B, but its inverse is not.)

Exercise 31. Let $X = \{a, b, c, d\}$ with strict preferences $a \succ b$, $a \succ c$, $b \succ d$, $c \succ d$, and b, c incomparable. (Take $\succsim$ to be the reflexive-transitive closure of these.) Show that no $u : X \to \mathbb{R}$ represents $\succsim$ in the sense of $x \succsim y \iff u(x) \geq u(y)$.

Exercise 32. Compare $(\mathbb{R}^2, \preceq^L)$ (lexicographic) and $(\mathbb{R}^2, \preceq^P)$ (componentwise / Pareto):

1. For each, state whether it is a partial order, a total order, or only a preorder.
2. For each, identify the supremum of $\{(0, 0), (1, 0), (0, 1)\}$, if it exists.
3. Pick one and explain in a sentence why a utility representation either does or does not exist.

References

Davey, B. A. and H. A. Priestley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press.

Halmos, Paul R. (1960). *Naive Set Theory*. Princeton: Van Nostrand.