Open and closed sets

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Last revised: 5 May 2026

1 Motivation

Many of the theorems we want to use about $\mathbb{R}^n$ are statements about the *shape* of a set rather than its size or measure. A continuous function on a closed bounded set attains its maximum; a continuous function on a connected set takes every intermediate value; a continuous function on a compact set is uniformly continuous; a fixed-point theorem applies to continuous self-maps of a compact convex set. Each of these is a topological fact: it says that some structural property of the domain (closed-and-boundedness, connectedness, compactness) carries over into the behavior of functions defined on it.

For political economy, the topological vocabulary is the entry point to most of the existence theorems that matter: existence of an optimum (Weierstrass extreme value theorem, on a compact set), existence of a Nash equilibrium (Brouwer or Kakutani fixed-point theorems, on compact convex sets), existence of a market-clearing price (Brouwer applied to an excess-demand simplex), existence of a median in a single-peaked-preferences setting (the median voter theorem leans on connectedness and continuity of upper-contour sets). The statements come later; this handout is the vocabulary they are stated in.

The setting throughout is $\mathbb{R}^n$ with the Euclidean distance.[1] The arguments transfer to general metric spaces with no real change.

2 $\mathbb{R}^n$, distance, open balls

Most political-economy models live in $\mathbb{R}^n$ for some n — outcomes for n agents, ideal points in n-dimensional ideology space, allocations across n goods, parameter vectors of an econometric model. Before we can ask whether a function on such a space is continuous, or whether a feasible set is compact, we need a notion of how close two points are. The Euclidean distance is the obvious candidate and the one we use throughout. The open ball — the set of points within a given distance of a base point — is the elementary building block out of which the rest of the topology gets assembled.

We have used $\mathbb{R}^n$ informally already. The formal definition is the Cartesian product:

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ factors}} = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}\}.$$

[1] The right level of generality is not $\mathbb{R}^n$ but *metric spaces* (sets with a distance function satisfying the triangle inequality and a few other axioms), or, more abstractly still, *topological spaces* (sets equipped with a chosen collection of subsets called "open," satisfying closure under finite intersection and arbitrary union). Almost everything in this handout transfers verbatim to metric spaces, and most of it transfers to topological spaces with appropriate adjustments. We stay in $\mathbb{R}^n$ because it is enough for the political-economy applications we care about, the proofs are concrete, and the geometric intuition is familiar from calculus. A separate handout on metric and topological spaces is not currently planned — when applications force the generalization (e.g., function spaces in dynamic models, weak topologies in measure theory), it can be picked up then.

Elements are written either as tuples $(x_1, \dots, x_n)$ or as boldface vectors $\boldsymbol{x}$ when the algebraic structure is in view. The Euclidean *norm* of $\boldsymbol{x}$ is

$$\|\boldsymbol{x}\| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

and the Euclidean *distance* between two points is $d(\boldsymbol{x}, \boldsymbol{y}) := \|\boldsymbol{x} - \boldsymbol{y}\|$. The function d satisfies $d(\boldsymbol{x}, \boldsymbol{y}) \geq 0$ with equality iff $\boldsymbol{x} = \boldsymbol{y}$; symmetry $d(\boldsymbol{x}, \boldsymbol{y}) = d(\boldsymbol{y}, \boldsymbol{x})$; and the triangle inequality $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z})$. These three properties together say d is a *metric* on $\mathbb{R}^n$, and they are all we will use — the rest of the handout is essentially a series of definitions in terms of d.

Definition 1. For $\boldsymbol{x} \in \mathbb{R}^n$ and $r > 0$, the *open ball* of radius r around $\boldsymbol{x}$ is

$$B_r(\boldsymbol{x}) := \{\boldsymbol{y} \in \mathbb{R}^n : d(\boldsymbol{x}, \boldsymbol{y}) < r\}.$$

In $\mathbb{R}$ $(n = 1)$, $B_r(x) = (x - r, x + r)$ is just an open interval. In $\mathbb{R}^2$, $B_r(\boldsymbol{x})$ is the interior of a disk; in $\mathbb{R}^3$, the interior of a sphere; and so on. The strict inequality is what makes the ball "open" — the boundary of the disk is excluded. The corresponding *closed ball* $\overline{B}_r(\boldsymbol{x}) := \{\boldsymbol{y} : d(\boldsymbol{x}, \boldsymbol{y}) \leq r\}$ includes its boundary.

3 Open and closed sets

Why distinguish open sets from closed sets? The high-school answer — "open intervals don't include their endpoints, closed ones do" — is right but doesn't yet say why anyone should care. The substantive answer arrives in later sections: open sets are the natural domain on which continuity can be defined, and closed sets are exactly the sets that contain all their limits, which is what we'll need when proving that an optimizer or an equilibrium is in the feasible region rather than only at its edge. Most existence theorems in economic theory — the extreme value theorem for an optimum, Brouwer's theorem for a Nash equilibrium, market-clearing fixed-point arguments — work on sets that are at least closed, and often closed-and-bounded. For now the goal is to introduce both classes and their basic algebra.

The open balls are the building blocks. A set is *open* if every point in it has a little open ball around it that still lies in the set — if you can wiggle slightly in any direction without leaving.

Definition 2. A set $U \subseteq \mathbb{R}^n$ is *open* if for every $\boldsymbol{x} \in U$ there exists $r > 0$ with $B_r(\boldsymbol{x}) \subseteq U$. A set $F \subseteq \mathbb{R}^n$ is *closed* if its complement $\mathbb{R}^n \setminus F$ is open.

Open and closed are not opposites in the everyday sense: a set can be both, neither, or one without the other.

Example 3 (Open intervals are open; closed intervals are closed)**.** $(a, b) \subset \mathbb{R}$ is open: for $x \in (a, b)$, take $r = \min(x - a, b - x) > 0$, and $B_r(x) \subseteq (a, b)$. Its complement $(-\infty, a] \cup [b, \infty)$ is closed by definition. The closed interval $[a, b]$ is closed: a point $y \notin [a, b]$ has $r = \min(|y - a|, |y - b|) > 0$ keeping $B_r(y)$ on the same side, hence in the complement.

Example 4 (Half-open intervals are neither)**.** $[a, b) \subset \mathbb{R}$ is not open: the point a has no open ball entirely inside, since any $B_r(a) = (a - r, a + r)$ contains points less than a. It is not closed: its complement $(-\infty, a) \cup [b, \infty)$ is not open (the point b is a problem on the right).

Example 5 (Both open and closed)**.** The empty set $\emptyset$ and the whole space $\mathbb{R}^n$ are both open and closed. (The empty set vacuously satisfies the open-set condition; $\mathbb{R}^n$ trivially does. Each is the complement of the other, and the complement of an open set is closed.) These are the only two sets in $\mathbb{R}^n$ that are simultaneously open and closed — a fact that is the content of the connectedness of $\mathbb{R}^n$, which we will get to in §6.

Example 6 (Open balls are open)**.** $B_r(\boldsymbol{x})$ is open: for $\boldsymbol{y} \in B_r(\boldsymbol{x})$, set $\rho = r - d(\boldsymbol{x}, \boldsymbol{y}) > 0$. Then $B_\rho(\boldsymbol{y}) \subseteq B_r(\boldsymbol{x})$ by the triangle inequality: if $d(\boldsymbol{y}, \boldsymbol{z}) < \rho$, then $d(\boldsymbol{x}, \boldsymbol{z}) \leq d(\boldsymbol{x}, \boldsymbol{y}) + d(\boldsymbol{y}, \boldsymbol{z}) < d(\boldsymbol{x}, \boldsymbol{y}) + \rho = r$.

The collection of open sets in $\mathbb{R}^n$ has clean closure properties.

Proposition 7. *The open subsets of* $\mathbb{R}^n$ *satisfy:*

1. $\emptyset$ *and* $\mathbb{R}^n$ *are open.*
2. *Any union of open sets is open.*
3. *Any* finite *intersection of open sets is open.*

By complementation, the closed subsets satisfy: $\emptyset$ *and* $\mathbb{R}^n$ *are closed; any intersection of closed sets is closed; any finite union of closed sets is closed.*

Proof. (2) Let $U = \bigcup_{i \in I} U_i$ with each U_i open. For $\boldsymbol{x} \in U$, $\boldsymbol{x} \in U_i$ for some i, and there is $r > 0$ with $B_r(\boldsymbol{x}) \subseteq U_i \subseteq U$. (3) Let $U = U_1 \cap \cdots \cap U_k$ with each U_i open. For $\boldsymbol{x} \in U$, choose $r_i > 0$ with $B_{r_i}(\boldsymbol{x}) \subseteq U_i$ for each i, and take $r = \min(r_1, \ldots, r_k) > 0$; then $B_r(\boldsymbol{x}) \subseteq \bigcap U_i = U$. The closed-set claims follow from De Morgan's laws applied to (2) and (3). □

The finiteness in (3) is essential: an arbitrary intersection of open sets need not be open. The standard counter-example: $\bigcap_{n \geq 1}(-1/n, 1/n) = \{0\}$, a singleton, which is closed but not open.

4 Closure, interior, boundary, accumulation points

Most sets that arise in political-economy modeling are not open or closed by accident; they are built out of constraints (linear inequalities for budgets, weak vs. strict majority rules for winning coalitions, indifference equalities for preference relations) that mix open and closed pieces. The Pareto-improving region of an allocation, the strict winning region of a coalition, the indifference set of a continuous preference, the strictly preferred set of a utility function — each has an interior (where the property holds with room to spare), a closure (the smallest closed set containing it), and a boundary (where strict and non-strict diverge). The substantive content of a set often lives in one of those parts in particular: the boundary, for example, is where ties happen, and ties are typically where the existence theorems wobble. The four constructions of this section are the standard vocabulary for talking about which part of a set is which.

For an arbitrary set $A \subseteq \mathbb{R}^n$, four derived sets capture different aspects of how A sits inside the ambient space.

Definition 8. Let $A \subseteq \mathbb{R}^n$.

- The *interior* of A, written A° or $\operatorname{int}(A)$, is the set of $\boldsymbol{x} \in A$ such that some $B_r(\boldsymbol{x}) \subseteq A$. Equivalently, A° is the largest open set contained in A.
- The *closure* of A, written $\overline{A}$ or $\operatorname{cl}(A)$, is the set of $\boldsymbol{x} \in \mathbb{R}^n$ such that every $B_r(\boldsymbol{x})$ meets A. Equivalently, $\overline{A}$ is the smallest closed set containing A.
- The *boundary* of A, written ∂A, is $\overline{A} \setminus A^\circ$.
- A point $\boldsymbol{x} \in \mathbb{R}^n$ is an *accumulation point* (or *limit point*) of A if every $B_r(\boldsymbol{x})$ contains a point of A *other than $\boldsymbol{x}$ itself.* The set of accumulation points is denoted A'.

The four notions interlock cleanly:

Proposition 9. *For any $A \subseteq \mathbb{R}^n$:*

1. *A is open iff $A = A^\circ$.*
2. *A is closed iff $A = \overline{A}$ iff $A' \subseteq A$.*
3. *$\overline{A} = A \cup A'$.*
4. *$\partial A = \overline{A} \cap \overline{\mathbb{R}^n \setminus A}$ (so ∂A is itself always closed).*

We omit the proofs — they are direct unwinding of the definitions, and the reader who has not done so should write at least (2) out by hand.

Example 10 (Closure, interior, and boundary of (a, b))**.** $\overline{(a,b)} = [a, b]$, $(a, b)^\circ = (a, b)$, $\partial(a, b) = \{a, b\}$.

Example 11 (Closure, interior, boundary of $\mathbb{Q}$ in $\mathbb{R}$)**.** $\overline{\mathbb{Q}} = \mathbb{R}$ (every real has rationals arbitrarily close), $\mathbb{Q}^\circ = \emptyset$ (any open ball $B_r(q)$ around a rational contains irrationals, so no open ball lies inside $\mathbb{Q}$), $\partial\mathbb{Q} = \mathbb{R}$. So $\mathbb{Q}$ has empty interior and full closure: it is "everywhere and nowhere" inside $\mathbb{R}$.

Example 12 (Closure of a finite set)**.** A finite subset $F = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k\} \subset \mathbb{R}^n$ is closed (as a finite union of closed singletons), and $F^\circ = \emptyset$ (as long as $n \geq 1$), so $\partial F = F$.

The connection with the previous handout: a sequence converges to a limit if and only if every open ball around the limit eventually contains the sequence. So accumulation points of a set — and hence closed-ness — can be characterized in terms of sequences.

Theorem 13 (Sequential characterization of closure)**.** *Let $A \subseteq \mathbb{R}^n$ and $\boldsymbol{x} \in \mathbb{R}^n$. Then $\boldsymbol{x} \in \overline{A}$ if and only if there exists a sequence $(\boldsymbol{a}_k)$ in A with $\boldsymbol{a}_k \to \boldsymbol{x}$. Consequently, A is closed iff every convergent sequence in A has its limit in A.*

Proof. ($\Leftarrow$) If $\boldsymbol{a}_k \to \boldsymbol{x}$ with $\boldsymbol{a}_k \in A$, then for any $r > 0$, eventually $\boldsymbol{a}_k \in B_r(\boldsymbol{x}) \cap A$, so $B_r(\boldsymbol{x})$ meets A, so $\boldsymbol{x} \in \overline{A}$.

($\Rightarrow$) If $\boldsymbol{x} \in \overline{A}$, every $B_{1/k}(\boldsymbol{x})$ meets A for $k = 1, 2, 3, \ldots$. Pick $\boldsymbol{a}_k \in B_{1/k}(\boldsymbol{x}) \cap A$. Then $d(\boldsymbol{a}_k, \boldsymbol{x}) < 1/k \to 0$, so $\boldsymbol{a}_k \to \boldsymbol{x}$.

The closed-set version follows: A is closed iff $A = \overline{A}$, and the displayed equivalence rewrites this as: A contains every limit of a sequence in A. ☐

The sequential characterization is what one mostly uses in practice: to check that a set A is closed, one usually picks a convergent sequence in A and shows its limit is in A, rather than working directly with the open-set complement.

5 Compactness

Almost every existence theorem in mathematical political economy has compactness as a hypothesis somewhere. An optimum exists because the feasible set is compact (the Weierstrass extreme value theorem). A Nash equilibrium exists because the strategy space is compact (Brouwer or Kakutani). A market-clearing price exists because the price simplex is compact. The intuition is that compactness is the topological abstraction of "finite enough that the proofs work" — the technical cousin of finiteness that lets one push around quantifiers, pull subsequences out of bounded sequences, and rule out runaway behavior at the edges of a set. On the line, the closed bounded interval $[a, b]$ is the prototypical compact set, and all of the calculus theorems that make $[a, b]$ a special domain — continuous functions attain extrema, are uniformly continuous, integrate cleanly — are theorems about compactness.

The official open-cover definition.

Definition 14. A set $K \subseteq \mathbb{R}^n$ is *compact* if every open cover of K has a finite subcover. Spelled out: whenever $\{U_i\}_{i \in I}$ is a collection of open sets with $K \subseteq \bigcup_{i \in I} U_i$, there exist finitely many indices $i_1, \ldots, i_k \in I$ with $K \subseteq U_{i_1} \cup \cdots \cup U_{i_k}$.

The definition is austere on first encounter. In $\mathbb{R}^n$ a more concrete characterization is available, due to Heine and Borel.

Theorem 15 (Heine–Borel)**.** *A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.*

Proof sketch. Compact $\Rightarrow$ closed and bounded. For boundedness: cover K by the open balls $\{B_n(\mathbf{0})\}_{n \geq 1}$; a finite subcover is contained in a single $B_n(\mathbf{0})$. For closedness: if $\boldsymbol{x} \notin K$, the open balls $\{\mathbb{R}^n \setminus \overline{B}_{1/n}(\boldsymbol{x})\}_{n \geq 1}$ cover K (every point of K is at positive distance from $\boldsymbol{x}$); a finite subcover, equivalent to a single such complement, separates $\boldsymbol{x}$ from K by a positive distance, so $\boldsymbol{x}$ has an open ball disjoint from K.

Closed and bounded $\Rightarrow$ compact. Bounded means $K \subseteq [-M, M]^n$ for some M. The closed cube $[-M, M]^n$ is compact (a non-trivial argument: bisect into 2^n sub-cubes, recurse on the one not finitely covered, get a nested sequence of cubes with shrinking diameter; the intersection point is in K but uncovered, contradicting that the cover was a cover). A closed subset of a compact set is compact (a routine argument: any open cover of K, augmented by the open complement of K, covers the whole compact ambient set). Combining: K closed and bounded $\Rightarrow$ K a closed subset of a compact set $\Rightarrow$ K compact. □

In a metric space, compactness is also equivalent to *sequential compactness* — every sequence has a convergent subsequence whose limit is in the set — and this is the formulation that connects directly to Bolzano–Weierstrass from the previous handout.

Theorem 16 (Sequential characterization of compactness in $\mathbb{R}^n$)**.** *A subset $K \subseteq \mathbb{R}^n$ is compact if and only if every sequence in K has a convergent subsequence whose limit is in K.*

Proof sketch. ($\Rightarrow$) If K is compact, then K is closed and bounded by Heine–Borel. Bolzano–Weierstrass gives a convergent subsequence; closedness of K plus the sequential characterization of closure puts the limit in K. ($\Leftarrow$) Every sequential limit lying in K gives closedness (Theorem 13). Boundedness: if K were unbounded, choose $\boldsymbol{x}_n \in K$ with $\|\boldsymbol{x}_n\| \geq n$; this sequence has no convergent subsequence (every subsequence is unbounded). So K is closed and bounded, hence compact by Heine–Borel. □

The three characterizations of compactness — open-cover, sequential, closed-and-bounded — are equivalent in $\mathbb{R}^n$, but the equivalences fall apart in greater generality, and it is worth being clear about where.[2]

The headline theorem about compactness — the one that makes most of the existence theorems in the rest of mathematics possible — requires continuity, and so we defer it to the continuity handout. Stated in advance: a continuous function on a compact set attains its maximum and minimum (the *extreme value theorem*). The compactness side of the proof is what we have set up in this section; the continuity side is what comes next.

6 Connectedness

Compactness is about "finite enough." Connectedness is about "in one piece" — and it is the topological reason continuous functions take all intermediate values, which is itself the topological reason the standard market-clearing argument works (excess demand is positive at low prices, negative at high ones, and continuous in between, so it must equal zero somewhere). For most political-economy applications, the relevant connected sets are convex sets — feasible regions defined by linear constraints, mixed-strategy simplices, budget sets, upper-contour sets of concave utility functions — and the proposition at the end of this section that convex implies connected is what does most of the work.

The formal definition of connectedness is by negation: a set is connected if it cannot be split into two open pieces.

Definition 17. A set $A \subseteq \mathbb{R}^n$ is *disconnected* if there exist open sets $U, V \subseteq \mathbb{R}^n$ with $A \cap U \neq \emptyset$, $A \cap V \neq \emptyset$, $A \subseteq U \cup V$, and $A \cap U \cap V = \emptyset$. The pair (U, V) is called a *separation* of A. The set A is *connected* if no such separation exists.

The negation is awkward but unavoidable: there is no positive "you can get from anywhere to anywhere" formulation that works directly in arbitrary topological spaces. (There is a positive formulation, *path-connectedness*, which is strictly stronger; for "nice" subsets of $\mathbb{R}^n$ — in particular for open subsets — the two coincide, and we will use the positive intuition freely below.)

In $\mathbb{R}$, connectedness coincides with the natural geometric idea: connected sets are exactly intervals.

Theorem 18. *A subset $A \subseteq \mathbb{R}$ is connected if and only if A is an interval (in the sense that whenever $a, b \in A$ and $a < c < b$, also $c \in A$).*

[2] The equivalence *open-cover compactness* $\Leftrightarrow$ *sequential compactness* holds in any metric space, not just $\mathbb{R}^n$. The equivalence *compactness* $\Leftrightarrow$ *closed-and-bounded* is much more parochial: it holds in $\mathbb{R}^n$ (Heine–Borel) but fails in essentially every infinite-dimensional space one cares about. The standard counter-example is the unit ball in an infinite-dimensional normed space (e.g., ℓ^2, the space of square-summable sequences): closed and bounded, but not compact, because the standard basis vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots$ form a bounded sequence with no convergent subsequence (consecutive basis vectors are at distance $\sqrt{2}$). This matters in political-economy applications that use infinite-dimensional spaces: a space of utility functions, a space of probability distributions, a space of allocations across an infinite agent type space. In these settings compactness has to be re-earned (e.g., via Helly's selection theorem for distributions, or via weak topologies that make bounded sets compact at the price of changing what convergence means). The takeaway: closed-and-bounded works as a definition of "finite enough" only because we are in $\mathbb{R}^n$. In general one must check the open-cover or sequential property directly. The compactness chapter in Rudin (1976) works through Heine–Borel and the basic equivalences carefully.

Proof sketch. ($\Rightarrow$, contrapositive) If A is not an interval, pick $a, b \in A$ with some $c \in (a, b) \setminus A$. The pair $U = (-\infty, c)$, $V = (c, \infty)$ separates A. ($\Leftarrow$) Suppose A is an interval and (U, V) is a separation, with $a \in A \cap U$ and $b \in A \cap V$, say $a < b$. Let $s = \sup\{x \in [a, b] : x \in U\}$. Then $s \in [a, b] \subseteq A$, so $s \in U$ or $s \in V$. If $s \in U$, then since U is open, some $B_r(s) \subseteq U$, and $s + r/2 > s$ is in $U \cap [a, b]$, contradicting that s is the supremum. If $s \in V$, then V is open and contains a neighborhood of s, contradicting that points just below s are in U. □

In $\mathbb{R}^n$ for $n \geq 2$ the situation is more interesting: a much wider variety of sets is connected, including, for example, every convex set.

Example 19 (Convex sets are connected)**.** A set $C \subseteq \mathbb{R}^n$ is *convex* if for every $\boldsymbol{x}, \boldsymbol{y} \in C$ and every $t \in [0, 1]$, the point $(1 - t)\boldsymbol{x} + t\boldsymbol{y}$ lies in C (the line segment from $\boldsymbol{x}$ to $\boldsymbol{y}$ stays in C). Convex sets are connected: any separation (U, V) would split $\boldsymbol{x} \in C \cap U$ from $\boldsymbol{y} \in C \cap V$, but the line-segment map $f(t) = (1 - t)\boldsymbol{x} + t\boldsymbol{y}$ pulls a separation in C back to a separation of $[0, 1]$ in $\mathbb{R}$, contradicting Theorem 18.

This example covers most of the political-economy applications: feasible sets defined by linear constraints, simplices of mixed strategies, budget sets, upper-contour sets of concave utility functions — all are convex, hence connected.

7 What's next

The natural successor is the continuity handout. Continuity is what binds together everything we have set up. The headline theorems:

- *Sequential characterization of continuity*: f is continuous at $\boldsymbol{x}$ iff $\boldsymbol{x}_n \to \boldsymbol{x}$ implies $f(\boldsymbol{x}_n) \to f(\boldsymbol{x})$. (Connects §4 of this handout to the sequence-and-limit machinery of the previous one.)
- *Extreme value theorem*: a continuous real-valued function on a compact set attains its maximum and minimum. (The reason we care about compactness.)
- *Intermediate value theorem*: a continuous real-valued function on a connected set takes every value between any two it takes. (The reason we care about connectedness.)
- *Uniform continuity*: a continuous function on a compact set is automatically uniformly continuous. (Sharper-than-pointwise, free for compact domains.)

The probability-and-measure cluster will eventually pick up topology in a different direction: the Borel σ-algebra is generated by the open sets, and the basic measurable-set machinery of probability runs on top of the topological vocabulary developed here.

For a full treatment of the topology of $\mathbb{R}^n$ at this level, see Rudin (1976) (chapter 2) or Abbott (2015) (chapter 3); for a more abstract development covering general metric spaces, see Munkres (2000).

8 Exercises

Exercise 20. For each of the following subsets of $\mathbb{R}$, determine whether it is open, closed, both, or neither: $(0,1)$; $[0,1]$; $[0,1)$; $\{1/n : n \geq 1\}$; $\{0\} \cup \{1/n : n \geq 1\}$; $\mathbb{Z}$; $\mathbb{R} \setminus \mathbb{Z}$.

Exercise 21. Show that the countable intersection of open sets in $\mathbb{R}$ need not be open: the example in §3 shows $\bigcap_n (-1/n, 1/n) = \{0\}$, which is closed but not open. Construct a countable union of closed sets that is not closed. (Hint: the rationals.)

Exercise 22. Let $A \subseteq \mathbb{R}^n$. Prove $\overline{A} = A \cup A'$ (the closure equals the set together with its accumulation points), and conclude that A is closed iff $A' \subseteq A$.

Exercise 23. Compute A°, $\overline{A}$, and ∂A for each of the following $A \subseteq \mathbb{R}^2$:

1. $A = \{(x, y) : x^2 + y^2 < 1\}$ (the open unit disk).
2. $A = \{(x, y) : x^2 + y^2 = 1\}$ (the unit circle).
3. $A = \{(x, y) : x \in \mathbb{Q}, y \in \mathbb{Q}\}$ (the rational lattice points).

Exercise 24 (Pareto frontier on the boundary)**.** Let $U \subseteq \mathbb{R}^n$ be a closed bounded set of utility profiles, with the agreement that $\boldsymbol{x} \preceq^P \boldsymbol{y}$ means $x_i \leq y_i$ for every i (componentwise / weak Pareto dominance). Show that any Pareto-maximal element (any $\boldsymbol{x} \in U$ with no $\boldsymbol{y} \in U$ satisfying $\boldsymbol{y} \succ^P \boldsymbol{x}$) lies in ∂U. (Hint: if $\boldsymbol{x} \in U^\circ$, some $B_r(\boldsymbol{x}) \subseteq U$, and within that ball you can move strictly upward in every coordinate.)

Exercise 25 (Upper-contour sets and single-peaked preferences)**.** Let $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ and $u(\boldsymbol{x}) := -\|\boldsymbol{x} - \bar{\boldsymbol{x}}\|^2$ (Euclidean quadratic-loss preferences with bliss point $\bar{\boldsymbol{x}}$). For any $c \leq 0$, the upper-contour set is $\{\boldsymbol{x} : u(\boldsymbol{x}) \geq c\} = \overline{B}_{\sqrt{-c}}(\bar{\boldsymbol{x}})$. Verify that this set is closed, convex (hence connected), and equal to $\mathbb{R}^n$ exactly when $c \leq -\|\bar{\boldsymbol{x}}\|^2$ – in particular, single-peaked along every line through $\bar{\boldsymbol{x}}$. The topological scaffolding here is what underlies the median voter theorem in higher dimensions.

Exercise 26. Use Theorem 16 to prove the following: if $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of nonempty compact sets in $\mathbb{R}^n$, then $\bigcap_n K_n$ is nonempty (the *Cantor intersection theorem*). Where in your proof does compactness enter, and where would the argument fail if the K_n were merely closed?

Exercise 27 (Compact strategy spaces)**.** A common modeling convention is to take the strategy space of a game to be a compact subset of $\mathbb{R}^n$ (e.g., $[0,1]^k$ for mixed strategies in a k-action game, or a closed bounded "policy space" for ideal-point models). Without proving any equilibrium-existence theorems yet (we will get to these in the continuity handout and beyond), explain in two or three sentences why compactness is the operative property: what would go wrong, intuitively, if the strategy space were closed but unbounded? If open and bounded?

Exercise 28. Show that $\partial A = \partial(\mathbb{R}^n \setminus A)$ for any $A \subseteq \mathbb{R}^n$ (a set and its complement share a boundary). Conclude that the boundary is symmetric in a sense the interior and closure are not.

Exercise 29 (Winning regions in 2D ideology space)**.** Two candidates L and R have ideal points $\boldsymbol{x}_L, \boldsymbol{x}_R \in \mathbb{R}^2$, and a voter at $\boldsymbol{v} \in \mathbb{R}^2$ votes for whichever candidate is closer in Euclidean distance. Describe the *winning region* of L — the set of voter positions that vote for L — as a subset of $\mathbb{R}^2$. Is it open, closed, both, neither? What is its boundary, and in what circumstance does this boundary matter for an election outcome? (Hint: it is one of the two open half-planes determined by the perpendicular bisector of the segment $\overline{\boldsymbol{x}_L \boldsymbol{x}_R}$. The boundary is the bisector itself, a measure-zero set, which would matter only if a positive mass of voters lay exactly on it; this corresponds to ties.)

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