Natural transformations and the Yoneda lemma

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1 Motivation

The previous handout left two pieces of business unfinished. First, the definition of equivalence of categories invoked an "isomorphism of functors" that we had no language for. Second, the hom-functors $\mathrm{Hom}(A, -)$ were introduced as central but their centrality was asserted, not displayed. The handout now in your hands fixes both.

The first fix is the right notion of map between functors. A natural transformation assigns, to each object, a morphism between the two functors' images, in a way compatible with the source category's morphisms. Functors between two fixed categories form a category in their own right — the functor category — and "isomorphism of functors" is just isomorphism in that category. With that vocabulary, equivalence of categories from the previous handout becomes precise on the nose.

The second fix is the Yoneda lemma, the working theorem of this cluster. The statement is short. For any locally small $\mathcal{C}$, any object $A \in \mathcal{C}$, and any functor $F : \mathcal{C} \to \mathbf{Set}$, the natural transformations $\mathrm{Hom}(A, -) \Rightarrow F$ are in bijection with the elements of $F(A)$. The slogans the lemma supports include "an object is determined by its maps in," "the Hom-functor is the freest thing one can build out of an object," and "a functor to **Set** is essentially a description of how each object's elements transform under morphisms." Each of these is reasonable. None is a substitute for working through the lemma.

For a political-economy reader, the immediate payoff is modest: Yoneda is not directly applied in working political-economy papers very often. The cumulative payoff is real. Almost every category-flavored argument the reader will encounter — in the structural treatments of stochastic processes, in the model theory of social choice (the previous handout introduced the semantic view), in the categorical packaging of choice and decision data that the literature on revealed preference flirts with — runs through some incarnation of Yoneda. The slogan "an object is determined by its maps in" is also a near-twin, in spirit, of the revealed-preference slogan "an alternative is determined by the menus on which it is chosen." We will sound the analogy at the end of the handout but not over-claim it.

2 Natural transformations

A functor $F : \mathcal{C} \to \mathcal{D}$ moves a whole category at once. Given two such functors, the right notion of "how do they compare" is one that tracks them *point by point* (one comparison per object of $\mathcal{C}$) but *coherently* (the comparisons match the morphism data). That is what natural transformations do.

Definition 1. Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\eta : F \Rightarrow G$ assigns to each object $A \in \mathcal{C}$ a morphism $\eta_A : F(A) \to G(A)$ in $\mathcal{D}$, called the *component of* η *at* A, such that for every morphism $f : A \to B$ in $\mathcal{C}$ the *naturality square* commutes: $G(f) \circ \eta_A = \eta_B \circ F(f)$.

$$
\begin{array}{ccc}
F(A) & \xrightarrow{F(f)} & F(B) \\
{\scriptstyle \eta_A}\big\downarrow & & \big\downarrow{\scriptstyle \eta_B} \\
G(A) & \xrightarrow[G(f)]{} & G(B)
\end{array}
$$

A natural transformation is a *natural isomorphism* if every component η_A is iso in $\mathcal{D}$. In that case the componentwise inverses η_A^{-1} themselves assemble into a natural transformation $\eta^{-1} : G \Rightarrow F$ (Exercise 15), so a natural isomorphism is exactly an iso in the functor category we will define in §3.

The naturality square is the heart of the definition. "Pointwise compatibility" would be just having a morphism $\eta_A : F(A) \to G(A)$ for each A, with no further constraint. "Naturality" is the additional demand that those pointwise comparisons be compatible with the way both functors transport morphisms.

Example 2 (Pointwise inequality)**.** Let $(A, \preceq)$ and $(B, \preceq)$ be preorders, viewed as thin categories (the thin-category construction of the previous handout). Let $F, G : (A, \preceq) \to (B, \preceq)$ be order-preserving functions, viewed as functors. A natural transformation $\eta : F \Rightarrow G$ assigns to each $a \in A$ a morphism $\eta_a : F(a) \to G(a)$ in $(B, \preceq)$. Such a morphism exists iff $F(a) \preceq G(a)$, in which case it is unique (each Hom-set has at most one element). Naturality squares commute automatically (any equation between morphisms in a thin category holds, since each Hom-set has at most one element). So a natural transformation $F \Rightarrow G$ is exactly the pointwise inequality $F(a) \preceq G(a)$ for every $a \in A$. The functor category from $(A, \preceq)$ to $(B, \preceq)$ is itself a preorder — the *pointwise order* on order-preserving maps.

Example 3 (Double-dual on vector spaces)**.** Let **Vect** denote the category of real vector spaces and linear maps. The dual $V^* := \mathrm{Hom}_{\mathbf{Vect}}(V, \mathbb{R})$ is a contravariant functor; the double-dual V^{**} is covariant. There is a natural transformation $\eta : \mathrm{id}_{\mathbf{Vect}} \Rightarrow (-)^{**}$ given by $\eta_V(v)(\phi) = \phi(v)$ for $\phi \in V^*$, sometimes called the *evaluation* natural transformation. Naturality says: for any linear $f : V \to W$, the square commutes:

$$f^{**} \circ \eta_V = \eta_W \circ f.$$

On finite-dimensional spaces η_V is an iso, and the natural-transformation framing is exactly what one means by saying "the iso $V \cong V^{**}$ is canonical, no choice of basis required." Without the naturality, there are isos $V \cong V^*$ in finite dimensions too, but those depend on a choice of basis and do not assemble into a natural transformation $\mathrm{id} \Rightarrow (-)^*$. Naturality is the precise sense in which double-dual is canonical and single-dual is not.

Example 4 (Power-set inclusion)**.** Let $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ be the covariant power-set functor of the previous handout. The singleton-inclusion $\eta_A : A \to \mathcal{P}(A)$, $a \mapsto \{a\}$, is the component of a natural transformation $\mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$: naturality says, for $f : A \to B$, that $\mathcal{P}(f)(\{a\}) = \{f(a)\}$, which is the definition of $\mathcal{P}(f)$ on singletons.

Example 5 (Voting rules and pointwise comparison)**.** Let $w, w' : \mathcal{N} \to \mathbf{2}$ be two voting rules on a finite legislature, viewed as functors out of the coalition lattice (the voting-rule-as-functor example of the previous handout). A natural transformation $w \Rightarrow w'$ is the same as the pointwise inequality $w(\sigma) \leq w'(\sigma)$ for every coalition σ — by Example 2, the functor category is the pointwise order. So "w' is a more permissive voting rule than w" is exactly "there is a natural transformation $w \Rightarrow w'$." This is the structural rendering of the comparison; the structural framing makes it obvious that any two voting rules are either comparable (by pointwise dominance) or not, and that the comparison either holds for all coalitions or fails to be a natural transformation.

3 Functor categories

Once natural transformations are in hand, functors between two fixed categories form a category. The objects are functors, the morphisms are natural transformations, and composition is *vertical*: stack one natural transformation on top of another.

Definition 6. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The *functor category* $[\mathcal{C}, \mathcal{D}]$ (also written $\mathcal{D}^{\mathcal{C}}$ or $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$) has:

- objects: functors $\mathcal{C} \to \mathcal{D}$;
- morphisms: natural transformations between such functors;
- composition: *vertical composition* of natural transformations. If $\eta : F \Rightarrow G$ and $\epsilon : G \Rightarrow H$, the composite $\epsilon \circ \eta : F \Rightarrow H$ has components $(\epsilon \circ \eta)_A := \epsilon_A \circ \eta_A$;
- identity: the *identity natural transformation* id_F with components $\mathrm{id}_{F(A)}$.

The category axioms are inherited from $\mathcal{D}$ componentwise.

The functor category resolves the loose end of the previous handout. Equivalence of categories was defined as the existence of $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$, for some sense of $\cong$ at the level of functors. That sense is now precise: $\cong$ is isomorphism in the functor categories $[\mathcal{C}, \mathcal{C}]$ and $[\mathcal{D}, \mathcal{D}]$, i.e., natural isomorphism. We can finally state the equivalence theorem from the previous handout cleanly.

Theorem 7 (Equivalence characterization, completed)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories iff F is fully faithful and essentially surjective.*

The proof is sketched in Exercise 22 after the Yoneda lemma is in hand.

A special case worth flagging: when $\mathcal{D} = \mathbf{Set}$, the functor category $[\mathcal{C}, \mathbf{Set}]$ is sometimes called the category of *copresheaves* on $\mathcal{C}$. The contravariant version, $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$, is the category of *presheaves* on $\mathcal{C}$. Presheaves are the natural target for the Yoneda embedding in §5; the choice of contravariant convention is essentially a choice of which side of the duality to highlight, with presheaves being the more standard one.

4 Hom-functors and representable functors

The previous handout introduced two Hom-functors associated with each object A in a locally small $\mathcal{C}$: the covariant $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$ (post-composition) and the contravariant $\mathrm{Hom}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ (pre-composition). The functors are easy to write down. What makes them central is that any functor to $\mathbf{Set}$ that "looks like" one of these has a special status.

Definition 8. A functor $F : \mathcal{C} \to \mathbf{Set}$ is *representable* if there exists an object $A \in \mathcal{C}$ and a natural isomorphism $F \cong \mathrm{Hom}(A, -)$. The object A is then called a *representing object* for F. Dually, a presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ is representable if $F \cong \mathrm{Hom}(-, A)$ for some A.

Example 9 (Identity is representable)**.** The identity functor $\mathrm{id}_{\mathbf{Set}} : \mathbf{Set} \to \mathbf{Set}$ is representable, with representing object $\{*\}$. The natural iso is $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, B) \cong B$, $f \mapsto f(*)$. Naturality is immediate: for $g : B \to B'$, $g \circ f$ on the left corresponds to $g(f(*))$ on the right.

Example 10 (Forgetful functors are often representable)**.** The forgetful functor $U : \mathbf{Pos} \to \mathbf{Set}$ is representable. The representing object is the singleton poset $\{*\}$ (one element, $\preceq$ trivial). The natural iso is $\mathrm{Hom}_{\mathbf{Pos}}(\{*\}, P) \cong U(P)$, $f \mapsto f(*)$. Same recipe as **Set**: a function out of a one-element set is just an element, and order-preservation from a discrete one-element source is vacuous, so order-preserving maps and underlying-set elements coincide. The same recipe representable many other forgetful functors — $U : \mathbf{Top} \to \mathbf{Set}$ (representing object: the one-point space), $U : \mathbf{Grp} \to \mathbf{Set}$ (representing object: the free group on one generator, $\mathbb{Z}$), and so on. The pattern "forgetful functor is representable by the free object on one generator" is a general one and is the first hint of the adjunction between forgetful and free functors.

Example 11 (Subobjects are co-representable)**.** The contravariant functor $\mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$ sending A to $\mathcal{P}(A)$ (the set of subsets) and $f : A \to B$ to the preimage map $\mathcal{P}(f) : \mathcal{P}(B) \to \mathcal{P}(A)$, $S \mapsto f^{-1}(S)$, is representable: it is naturally isomorphic to $\mathrm{Hom}(-, \{0,1\})$, with the iso sending a subset $S \subseteq A$ to its indicator function $\mathbf{1}_S : A \to \{0,1\}$. So $\{0,1\}$ is a representing object (in this case, the *subobject classifier*) for the contravariant power-set presheaf.

The pattern across these examples is the same: a functor is representable when it can be "computed" as the Hom-set into a single distinguished object. The Yoneda lemma is the deeper statement that this kind of computation is more pervasive than the examples might suggest.

5 The Yoneda lemma

The Yoneda lemma is a structural identity. It says that the natural transformations from a hom-functor $\mathrm{Hom}(A, -)$ into any other functor $F : \mathcal{C} \to \mathbf{Set}$ are in bijection with the elements of $F(A)$. The bijection is utterly explicit, the proof is a few lines, and the consequences fill chapters of category theory texts.

Theorem 12 (Yoneda lemma)**.** *Let $\mathcal{C}$ be a locally small category, $A \in \mathrm{Ob}(\mathcal{C})$, and $F : \mathcal{C} \to \mathbf{Set}$ a functor. The map*

$$\Phi : \mathrm{Nat}(\mathrm{Hom}(A, -),\, F) \longrightarrow F(A), \qquad \Phi(\eta) = \eta_A(\mathrm{id}_A)$$

is a bijection. Its inverse sends $x \in F(A)$ to the natural transformation $\eta^x : \mathrm{Hom}(A, -) \Rightarrow F$ with components

$$\eta^x_B : \mathrm{Hom}(A, B) \to F(B), \qquad \eta^x_B(f) = F(f)(x).$$

Proof. **Well-definedness of the inverse.** Given $x \in F(A)$, define η^x_B as in the statement. We must check naturality: for $g : B \to B'$, the square

$$\begin{array}{ccc} \mathrm{Hom}(A,B) & \xrightarrow{(g\circ -)} & \mathrm{Hom}(A,B') \\ \downarrow \eta^x_B & & \downarrow \eta^x_{B'} \\ F(B) & \xrightarrow{F(g)} & F(B') \end{array}$$

commutes. Pick $f \in \mathrm{Hom}(A, B)$. Going right-then-down: $\eta^x_{B'}(g \circ f) = F(g \circ f)(x) = F(g)(F(f)(x))$ by functoriality of F. Going down-then-right: $F(g)(\eta^x_B(f)) = F(g)(F(f)(x))$. The two agree.

Inverse on the left ($\Phi(\eta^x) = x$). By construction, $\Phi(\eta^x) = \eta^x_A(\mathrm{id}_A) = F(\mathrm{id}_A)(x) = \mathrm{id}_{F(A)}(x) = x$, using $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$.

Inverse on the right ($\eta^{\Phi(\eta)} = \eta$). Set $x := \Phi(\eta) = \eta_A(\mathrm{id}_A)$ and pick $f \in \mathrm{Hom}(A, B)$. We compute:

$$\eta^x_B(f) = F(f)(x) = F(f)(\eta_A(\mathrm{id}_A)).$$

Naturality of η at $f : A \to B$ says $F(f) \circ \eta_A = \eta_B \circ \mathrm{Hom}(A, f)$, where $\mathrm{Hom}(A, f)$ is post-composition with f. Apply both sides to id_A: $F(f)(\eta_A(\mathrm{id}_A)) = \eta_B(f \circ \mathrm{id}_A) = \eta_B(f)$. So $\eta^x_B(f) = \eta_B(f)$, which is the equality of natural transformations componentwise.

The bijection is also natural in both A and F; we omit that check, which uses the same mechanical naturality-square unpacking. (See Mac Lane (1998, Theorem III.2.1) or Riehl (2017, §2.2) for the full statement.) □

The lemma is at first sight a piece of bookkeeping. The slogans it supports are not.[1]

6 The Yoneda embedding

Apply the Yoneda lemma with $F = \mathrm{Hom}(B, -)$ for some other object B. Then $F(A) = \mathrm{Hom}(B, A)$, and the lemma gives

$$\mathrm{Nat}(\mathrm{Hom}(A, -), \mathrm{Hom}(B, -)) \cong \mathrm{Hom}(B, A).$$

Equivalently, using the contravariant form,

$$\mathrm{Nat}(\mathrm{Hom}(-, A), \mathrm{Hom}(-, B)) \cong \mathrm{Hom}(A, B).$$

The right-hand side is the morphism set in $\mathcal{C}$. The left-hand side is the morphism set between two presheaves on $\mathcal{C}$. So the morphisms in $\mathcal{C}$ *are* the natural transformations between the corresponding representable presheaves.

This says the assignment $A \mapsto \mathrm{Hom}(-, A)$ extends to a functor

$$y : \mathcal{C} \longrightarrow [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$$

(the *Yoneda embedding*) that is fully faithful: the morphism-set bijection above is *the* bijection $\mathcal{C}(A, B) \cong [\mathcal{C}^{\mathrm{op}}, \mathbf{Set}](\mathrm{Hom}(-, A), \mathrm{Hom}(-, B))$ that fully faithfulness requires. So $\mathcal{C}$ embeds into its presheaf category, and the embedding preserves and reflects morphisms.

Corollary 13 (Yoneda for objects)**.** *$A \cong B$ in $\mathcal{C}$ iff $\mathrm{Hom}(-, A) \cong \mathrm{Hom}(-, B)$ as presheaves.*

This is the formal version of "an object is determined by its maps in." If two objects support naturally-isomorphic Hom-data, they are themselves isomorphic. The converse is also immediate: an iso $A \cong B$ induces a natural iso $\mathrm{Hom}(-, A) \cong \mathrm{Hom}(-, B)$ by post-composition.

[1] The slogans are worth taking seriously, but each requires a small amount of unpacking to land. "A natural transformation out of $\mathrm{Hom}(A, -)$ is determined by where it sends id_A" is a literal reading of Φ. "Elements of $F(A)$ are the same as natural transformations $\mathrm{Hom}(A, -) \Rightarrow F$" is the bijection Φ^{-1}, but it relies on the convention that "elements of $F(A)$" includes any element of any set in the image of F. "A knows everything F does at A via the family of Hom-sets" is the philosophical reading, and the one that the Yoneda embedding (next section) makes precise. Each of these is right; none is the lemma. The lemma is the bijection, with the elements-as-natural-transformations correspondence being its content. The contravariant version of Yoneda is the formally identical statement with $\mathcal{C}^{\mathrm{op}}$ in place of $\mathcal{C}$ throughout: $\mathrm{Nat}(\mathrm{Hom}(-, A), F) \cong F(A)$ for any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$. This contravariant form is the more frequently invoked one in algebraic geometry, where "presheaf on $\mathcal{C}$" is a daily-use object.

The Yoneda embedding is one of the structurally cleanest theorems in mathematics. Almost every category-theoretic reasoning step that "transfers" a property of $\mathcal{C}$ into the presheaf category, or vice versa, runs through it. In algebraic geometry it underwrites the functor-of-points view of schemes (a scheme is determined by its functor of points, i.e., by what it represents); in algebraic topology it powers homotopy theory's use of representable functors as cohomology theories; in logic it is the bridge between syntax and semantics for many logical systems.

Example 14 (A revealed-preference echo, suggestive only)**.** A persistent slogan in the choice-theoretic tradition is that an alternative is determined by the choices it appears in: a banana is whatever-you-pick-when-the-menu-is-banana, apple together with whatever-you-pick-when-the-menu-is-banana, orange and so on. Under suitable consistency conditions (the Houthakker / WARP / SARP / GARP family from the upcoming decision-theory cluster), this is enough to recover the underlying preference relation, and so to recover the alternative's role. The structural shape of this argument is the same as Yoneda's: an object is determined by its incoming-arrows-from-test-objects. The analogy is a real one — the slogan is in both cases "object recognized by its relations to others" — but the strict analogy needs a category whose objects are alternatives and whose morphisms are choice-relevant relations, with a presheaf $\mathrm{Hom}(-, a) : (\text{menus})^{\mathrm{op}} \to \mathbf{Set}$ recording the menus on which a is chosen. Setting that up takes more care than we will give it here, and in particular the choice functions of revealed-preference theory are not natural transformations in the strict sense (they are not compatible with menu-restriction in general). The reason for flagging the analogy is honest: when a working political economist hears "an object is what it is by virtue of its maps in," the revealed-preference slogan is the closest cousin she has met. The categorical statement is the more general one; the choice-theoretic statement is the same shape under a strong consistency condition.

7 What's next

This handout closes the introductory category-theory cluster. Three strands extend it:

- *Limits and colimits.* The universal-property style from the previous handout's products and coproducts generalizes: a *limit* of a diagram is a universal cone over it; a *colimit* is a universal cocone. Equalizers, pullbacks, products as a special case, and the dual forms (coequalizers, pushouts, coproducts) are all instances. The Yoneda lemma is the structural reason for the well-behavedness of representable functors under limits.
- *Adjoint functors.* The relationship between forgetful and free functors — $U : \mathbf{Pos} \to \mathbf{Set}$ paired with the discrete-poset functor in the other direction; $U : \mathbf{Grp} \to \mathbf{Set}$ paired with the free-group functor; many others — is captured by the notion of an *adjunction*: a natural isomorphism $\mathrm{Hom}_{\mathcal{D}}(F(A), B) \cong \mathrm{Hom}_{\mathcal{C}}(A, G(B))$ in both arguments. Yoneda is one of the standard tools for verifying adjunctions in practice.
- *Category theory in political economy.* The applications are scattered. The categorical packaging of stochastic processes (categories of probability spaces with measurable kernels as morphisms; functoriality of the conditional expectation under filtration refinement) is the most actively-developed corner. Categorical mechanism design, where allocation rules are functors and incentive compatibility is a naturality condition, is a smaller corner that has been explored in the theoretical-computer-science literature. The semantic-view-of-models perspective from the

model theory and modeling handout is itself a categorical move, given the Galois correspondence between $\mathrm{Mod}(T)$ and $\mathrm{Th}(\mathcal{K})$.

For the standard theorems, Mac Lane (1998, Chs. III, IV) cover natural transformations, the Yoneda lemma, and the construction of limits and colimits in classical detail. Awodey (2010) and Riehl (2017) are paced for first readers; Leinster (2014) is the shortest serious introduction.

8 Exercises

Exercise 15. Let $\eta : F \Rightarrow G$ be a natural transformation each of whose components η_A is an isomorphism in $\mathcal{D}$. Show that the componentwise inverses $\eta_A^{-1} : G(A) \to F(A)$ assemble into a natural transformation $\eta^{-1} : G \Rightarrow F$. (Hint: compose the naturality square for η on left and right with the appropriate inverses.) Conclude that natural isomorphisms are the same as isomorphisms in the functor category.

Exercise 16. Verify that vertical composition of natural transformations is associative and unital, completing the proof that $[\mathcal{C}, \mathcal{D}]$ is a category.

Exercise 17. Verify Example 4: the singleton-inclusion $\eta_A : A \to \mathcal{P}(A)$, $a \mapsto \{a\}$, is the component of a natural transformation $\mathrm{id}_{\mathbf{Set}} \Rightarrow \mathcal{P}$. Then exhibit a natural transformation $\mathcal{P} \Rightarrow \mathrm{id}_{\mathbf{Set}}$ in the other direction, or argue informally why none exists.

Exercise 18. (Continuing Example 5.) Let $\mathcal{N}$ be the coalition lattice of an n-member legislature, viewed as a thin category. Show that the functor category $[\mathcal{N}, \mathbf{2}]$ is itself a poset, with objects the monotone $\{0, 1\}$-valued functions on $\mathcal{P}(N)$ (i.e., the Boolean voting rules) and the order being pointwise comparison. Identify the top and bottom elements of this poset (the unanimity rule and the always-rejecting rule). Compare the standard order-theoretic phrasing "w' is more permissive than w" with the categorical phrasing "there is a natural transformation $w \Rightarrow w'$."

Exercise 19. Show that the constant functor $\Delta_D : \mathcal{C} \to \mathcal{D}$ at $D \in \mathcal{D}$ (Exercise 9 of the previous handout) is representable when $\mathcal{C}$ has a terminal object and D is the image of that terminal object. (Hint: if $1 \in \mathcal{C}$ is terminal, then $\mathrm{Hom}(1, A)$ is a singleton for every A, so $\mathrm{Hom}(1, -) \cong \Delta_{\{*\}}$ as functors $\mathcal{C} \to \mathbf{Set}$.)

Exercise 20. Prove the Yoneda lemma (Theorem 12) in your own words. Then state and prove the dual: $\mathrm{Nat}(\mathrm{Hom}(-, A), F) \cong F(A)$ for any presheaf $F : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ and any object A.

Exercise 21. Use Corollary 13 to prove: in $\mathbf{Set}$, two sets A and B are isomorphic iff $|A| = |B|$. (Hint: $\mathrm{Hom}_{\mathbf{Set}}(-, A) \cong \mathrm{Hom}_{\mathbf{Set}}(-, B)$ as presheaves on $\mathbf{Set}$ implies $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, A) \cong \mathrm{Hom}_{\mathbf{Set}}(\{*\}, B)$ as sets, which says $|A| = |B|$ via the standard bijection.) Conclude that this is a categorical reproof of the cardinality-determines-isomorphism theorem from the cardinality-and-infinity handout.

Exercise 22. Sketch the proof of Theorem 7: a functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories iff F is fully faithful and essentially surjective. (Outline: *Necessity* is the easier direction — if F, G are part of an equivalence, the natural-iso condition forces F to be fully faithful and essentially surjective. *Sufficiency* uses the axiom of choice to pick, for each $D \in \mathcal{D}$, an object $G(D) \in \mathcal{C}$ and an iso $\epsilon_D : F(G(D)) \cong D$; full faithfulness lifts morphisms in $\mathcal{D}$ uniquely to morphisms in $\mathcal{C}$, defining G on morphisms. The Yoneda lemma is invoked to argue that G is itself a functor.) Identify exactly where each hypothesis is used.

Exercise 23. Two preorders $(A, \preceq)$ and $(B, \preceq)$, viewed as thin categories, are equivalent iff their quotients by the symmetric part are order-isomorphic posets. Prove this. (Hint: in a thin category, two functors are naturally isomorphic iff they agree on objects up to the symmetric part; chase through the previous handout's definition of equivalence of categories.)

Exercise 24. The slogan "an object is what it is by virtue of its maps in" (Yoneda for objects, Corollary 13) and the slogan "an alternative is what it is by virtue of the menus in which it is chosen" (revealed preference) have the same shape but are not the same theorem. Identify, in two or three sentences, what in revealed preference plays the role of: (i) "object," (ii) "test object," (iii) "Hom-set," (iv) the natural-isomorphism-of-presheaves condition. Where does the analogy strain?

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