Naive set theory

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1 Motivation

Almost every formal claim in political economy is, at bottom, a claim about sets. The set of voters preferring one candidate to another. The set of strategy profiles in a game. The set of policies a coalition could enact. The set of states of the world a player thinks are possible. We describe these collections constantly, and we operate on them constantly — combining, restricting, intersecting, mapping one to another — so it is worth setting down a careful vocabulary before the political economy gets going.

The qualifier "naive" deserves a word. Working mathematicians treat sets the way you would expect: a set is a collection of things; two sets are equal when they have the same elements; you can collect things into sets in the obvious ways. Set theorists treat sets the way an axiomatic system says you should — and the reason is that the obvious approach has one subtle failure point (Russell's paradox) that a careful axiomatic system has to rule out. We will be naive throughout this handout: we use the obvious approach, and we close with a short discussion of why the obvious approach is not strictly correct and what the careful version does about it. For applied work the naive view is enough; the foundational repair affects nothing we will actually do.

2 Sets and basic vocabulary

The starting vocabulary is small, and most of it shows up in political-economy modeling unannounced. Whenever you say "the voters who supported the incumbent," "the coalitions that could pass the bill," or "the strategy profiles in which player 1 defects," you have already specified a subset of some larger set without being formal about what that means. This section pins down the symbols and conventions — $\subseteq$ for inclusion, $\emptyset$ for the empty set, $\mathcal{P}(A)$ for the collection of all subsets of A (the standard object housing "all possible coalitions") — and articulates the principle of *extensionality*: a set is determined by what it contains, and nothing else.

A *set* is a collection of objects, called *elements* or *members*. We write $x \in A$ for "x is an element of A," and $x \notin A$ for the negation.

Definition 1 (Extensionality)**.** Sets A and B are *equal*, written $A = B$, if they have the same elements: $x \in A$ if and only if $x \in B$ for every x.

Definition 2. A is a *subset* of B, written $A \subseteq B$, if every element of A is an element of B. We write $A \subsetneq B$ when $A \subseteq B$ and $A \neq B$ (A is a *proper subset*).

Equality and mutual inclusion are the same thing: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Most equality proofs about sets work this way — prove each inclusion separately.

Definition 3. The *empty set*, written $\emptyset$, is the unique set with no elements: $x \notin \emptyset$ for every x. The empty set is a subset of every set.

Definition 4. In any given context, the *universe* (or *universal set*) U is the set of all objects under consideration. Reasoning about subsets is always reasoning relative to a universe: "the set of voters who supported candidate X" presumes a universe of voters, and the universe is usually understood from context.

We specify sets in two main ways. By *listing*: $\{1, 2, 3\}$. By *set-builder notation*: $\{x \in U : P(x)\}$, read "the set of x in U such that $P(x)$ holds."[1]

Definition 5. The *power set* of A, written $\mathcal{P}(A)$, is the set of all subsets of A:

$$\mathcal{P}(A) = \{S : S \subseteq A\}.$$

Example 6. $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. There are $2^2 = 4$ subsets. More generally, a finite set with n elements has 2^n subsets.

Example 7 (Coalitions)**.** Let $N = \{1, 2, \ldots, n\}$ be a set of legislators. A *coalition* is a subset of N. The set of all possible coalitions is $\mathcal{P}(N)$, with 2^n elements (including the empty coalition and the grand coalition N). Cooperative-game-theoretic notions like the core, the Shapley value, and minimal winning coalitions are defined as objects sitting inside $\mathcal{P}(N)$ and satisfying further conditions.

We will refer freely throughout to the standard sets of numbers: $\mathbb{N} = \{0, 1, 2, \ldots\}$, the natural numbers; $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the integers; $\mathbb{Q}$, the rational numbers (ratios p/q of integers with $q \neq 0$); and $\mathbb{R}$, the real numbers. The reals are treated for now informally — as known from calculus — with the careful construction of $\mathbb{R}$ from $\mathbb{Q}$ deferred to a separate handout.

For real numbers $a < b$ we use the standard interval notation: $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ (closed); $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ (open); and analogously $[a, b)$ and $(a, b]$ for half-open intervals.

3 Set operations

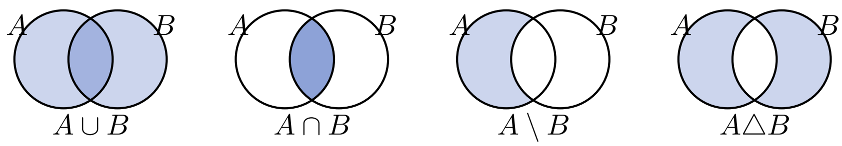
With the basic vocabulary in place, the natural next step is to combine and modify sets. The standard operations — union, intersection, complement, difference — correspond to "or," "and," "not," and "and-not" from propositional logic, and the Venn diagrams below are the standard pictures.

Definition 8. For sets A and B in some universe U:

$$\begin{aligned}
A \cup B &= \{x \in U : x \in A \text{ or } x \in B\} && (\textit{union}) \\
A \cap B &= \{x \in U : x \in A \text{ and } x \in B\} && (\textit{intersection}) \\
A \setminus B &= \{x \in U : x \in A \text{ and } x \notin B\} && (\textit{difference}) \\
A \triangle B &= (A \setminus B) \cup (B \setminus A) && (\textit{symmetric difference}) \\
A^c &= \{x \in U : x \notin A\} && (\textit{complement, relative to } \text{U})
\end{aligned}$$

Two sets are *disjoint* if $A \cap B = \emptyset$.

[1] Set-builder is restricted to "x in some already-existing set U" rather than unrestricted "all x such that $P(x)$." Unrestricted comprehension is what produces Russell's paradox; see §8. For applied work the restriction is invisible because the universe is always understood.



Proposition 9 (Standard set identities)**.** *For all sets A, B, C in a universe U:*

1. *Commutativity:* $A \cup B = B \cup A$ *and* $A \cap B = B \cap A$.
2. *Associativity:* $(A \cup B) \cup C = A \cup (B \cup C)$, *similarly for* $\cap$.
3. *Distribution:* $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, *and the same with* $\cap$ *and* $\cup$ *exchanged.*
4. *Identity:* $A \cup \emptyset = A$ *and* $A \cap U = A$.
5. *Absorption:* $A \cup (A \cap B) = A$ *and* $A \cap (A \cup B) = A$.
6. *De Morgan's laws:* $(A \cup B)^c = A^c \cap B^c$ *and* $(A \cap B)^c = A^c \cup B^c$.
7. *Complement:* $A \cup A^c = U$, $A \cap A^c = \emptyset$, *and* $(A^c)^c = A$.

Proof. Each follows from the corresponding propositional identity. For de Morgan: $x \in (A \cup B)^c$ iff $x \notin A \cup B$ iff $\neg(x \in A \vee x \in B)$ iff $x \notin A$ and $x \notin B$ iff $x \in A^c \cap B^c$. The other identities are similar. □

The structural similarity between the propositional identities (in the propositional-logic handout) and the set identities here is not a coincidence: $\vee, \wedge, \neg$ on truth values correspond exactly to $\cup, \cap, {}^c$ on subsets. The two identity lists are the same theorem twice, in different costumes.

4 Indexed operations

The union of two sets, three sets, ten sets is one thing, but in modeling we routinely need to combine entire families that are too large or too unbounded to enumerate. The voters who could plausibly support each of the fifty candidates in a primary; the policies that survive each of countably many rounds of iterated elimination of dominated strategies; the constituencies represented across the five hundred-odd districts of a national assembly. Each is a family $\{A_i\}_{i \in I}$ that needs to be combined all at once, and the indexed-union and indexed-intersection operations are the workhorses for doing so.

Definition 10. Let $\{A_i\}_{i \in I}$ be a family of subsets of U, indexed by a set I. Then:

$$\bigcup_{i \in I} A_i = \{x \in U : x \in A_i \text{ for some } i \in I\}$$
$$\bigcap_{i \in I} A_i = \{x \in U : x \in A_i \text{ for every } i \in I\}$$

When $I = \{1, 2, \ldots, n\}$ is finite we write $\bigcup_{i=1}^n A_i$ and $\bigcap_{i=1}^n A_i$. When $I = \mathbb{N}$ we write $\bigcup_{i=1}^\infty A_i$ similarly.

Example 11 (Coalition supporters)**.** Let V be the set of voters and let $C_1, \ldots, C_k$ be the candidates in an election. For each j, let $S_j \subseteq V$ be the set of voters who would support candidate C_j. Then:

- $\bigcup_{j=1}^{k} S_j$ is the set of voters who support *at least one* candidate.
- $\bigcap_{j=1}^{k} S_j$ is the set of voters who support *every* candidate.

The complement of the first — the voters who support no candidate — is, by de Morgan, $\bigcap_{j=1}^{k} S_j^c$.

Proposition 12 (Generalized de Morgan)**.** $\left(\bigcup_{i\in I} A_i\right)^c = \bigcap_{i\in I} A_i^c$ *and* $\left(\bigcap_{i\in I} A_i\right)^c = \bigcup_{i\in I} A_i^c$.

5 Cartesian products and tuples

A pair, a triple, an n-tuple: we need a way to glue elements together side by side, preserving order. Tuples and the Cartesian products that hold them are the fundamental machinery for talking about "a profile of one thing for each player" or "one input from set A and one from set B."

Definition 13. For elements a and b, the *ordered pair* (a, b) is determined by the property that $(a, b) = (a', b')$ if and only if $a = a'$ and $b = b'$.[2]

The ordered-pair notation (a, b) collides with the open-interval notation (a, b) from §2. Both are standard, both are unavoidable, and context will always make clear which is meant.

Definition 14. The *Cartesian product* of A and B is

$$A \times B = \{(a, b) : a \in A,\ b \in B\}.$$

For finite A and B, $|A \times B| = |A| \cdot |B|$. For sets $A_1, \ldots, A_n$,

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \ldots, a_n) : a_i \in A_i \text{ for each } i\},$$

with elements called *n-tuples*. We write A^n for $A \times A \times \cdots \times A$ (n copies).

Example 15 (Strategy profiles)**.** Let S_i be the strategy set of player i in an n-player game. A *strategy profile* is an element $(s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$. The full strategy space is the Cartesian product itself.

Example 16. $[0, 1] \times [0, 1] = [0, 1]^2$ is the unit square in $\mathbb{R}^2$.

We sometimes need a way to put two sets "side by side" so that elements stay distinguishable even when the originals overlap. The Cartesian-product machinery is exactly what's needed.

Definition 17. The *disjoint union* of A and B, written $A \sqcup B$, is the set

$$A \sqcup B = (A \times \{0\}) \cup (B \times \{1\}).$$

The tags 0 and 1 keep the elements of A and B distinguishable inside $A \sqcup B$ even if A and B share elements. For finite sets, $|A \sqcup B| = |A| + |B|$ always, in contrast to $|A \cup B| \leq |A| + |B|$ (with equality only when A and B are disjoint).

[2] There is a standard set-theoretic encoding of ordered pairs due to Kuratowski: $(a, b) := \{\{a\}, \{a, b\}\}$. The encoding ensures the defining property holds, and reduces "ordered pair" to a set-theoretic primitive. For applied work the encoding never needs to be unfolded; pairs are treated as primitive objects.

Example 18. $\{1,2\} \cup \{2,3\} = \{1,2,3\}$ has three elements — the shared 2 collapses. $\{1,2\} \sqcup \{2,3\} = \{(1,0),(2,0),(2,1),(3,1)\}$ has four: the 2 from the left and the 2 from the right are kept distinct as tagged copies.

The construction generalizes to a family of sets: for $\{A_i\}_{i \in I}$,

$$\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} \big(A_i \times \{i\}\big).$$

The indexed disjoint union is the right object when "putting collections together side by side" rather than "merging them into one" — an agent who appears in two of the A_i contributes two tagged copies.

6 Relations, equivalence, and partitions

A binary relation is a way of singling out which pairs of elements stand in some specified connection. "x is preferred to y." "x has the same party affiliation as y." "x defeats y in a head-to-head vote." Whatever the connection, the formal object is the same: a subset of $A \times A$.

Definition 19. A *binary relation* on a set A is a subset $R \subseteq A \times A$. We write $a \mathrel{R} b$ as a shorthand for $(a,b) \in R$.

Definition 20. A binary relation R on A is:

- *reflexive* if $a \mathrel{R} a$ for every $a \in A$;
- *symmetric* if $a \mathrel{R} b$ implies $b \mathrel{R} a$;
- *transitive* if $a \mathrel{R} b$ and $b \mathrel{R} c$ imply $a \mathrel{R} c$;
- *antisymmetric* if $a \mathrel{R} b$ and $b \mathrel{R} a$ imply $a = b$.

Different combinations of these properties give rise to different kinds of relations. Reflexive-antisymmetric-transitive defines a *partial order*, treated in the order theory handout. Reflexive-symmetric-transitive defines an *equivalence relation*, which is the case we develop here.

Definition 21. An *equivalence relation* on A is a binary relation that is reflexive, symmetric, and transitive. We typically write $\sim$ rather than a generic letter R. The *equivalence class* of $a \in A$ under $\sim$ is

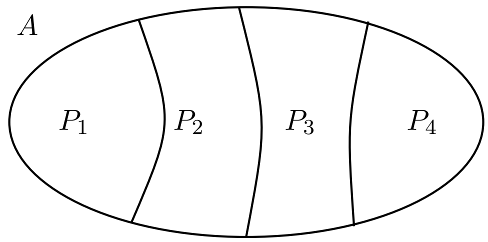
$$[a]_\sim = \{b \in A : a \sim b\}.$$

Example 22. On $\mathbb{Z}$, the relation $a \equiv b \pmod 5$ (i.e., $5 \mid (a-b)$) is an equivalence relation. Its equivalence classes are the five residue classes $[0],[1],[2],[3],[4]$, partitioning $\mathbb{Z}$ into five disjoint sets.

Example 23 (Voters by party)**.** Let V be a set of voters and define $v \sim v'$ if v and v' have the same party affiliation. Then $\sim$ is an equivalence relation, and its equivalence classes are the political parties.

Definition 24. A *partition* of A is a collection $\{P_i\}_{i \in I}$ of nonempty subsets of A such that

1. the P_i are pairwise disjoint: $P_i \cap P_j = \emptyset$ for $i \neq j$, and



2. their union is all of A: $\bigcup_{i \in I} P_i = A$.

Theorem 25 (Equivalence relations and partitions)**.** *Let A be a set.*

1. *Every equivalence relation $\sim$ on A induces a partition: the collection of equivalence classes $\{[a]_\sim : a \in A\}$.*
2. *Every partition $\{P_i\}_{i \in I}$ of A induces an equivalence relation: $a \sim b$ if and only if a and b lie in the same block.*

The two constructions are inverse: starting from $\sim$, taking equivalence classes, and then taking the same-block relation gives $\sim$ back; starting from a partition, taking the same-block relation, and then taking equivalence classes gives the partition back.

Proof. For (1): equivalence classes are nonempty (since $a \in [a]_\sim$ by reflexivity), they cover A (each a is in $[a]_\sim$), and they are either equal or disjoint. To see the last, suppose $[a] \cap [b] \neq \emptyset$, so some c lies in both; then $a \sim c$ and $c \sim b$, so $a \sim b$ by transitivity, hence $[a] = [b]$. For (2): the same-block relation is reflexive (each a is in its own block), symmetric ("in the same block" is symmetric), and transitive (if a and b share a block, and b and c share a block, the blocks must coincide). The inverse claim is immediate from the definitions. □

This is a key result. Whenever you classify the elements of a set — voters by party, candidates by ideology, alternatives by Pareto-equivalence — you are simultaneously specifying an equivalence relation. Equivalence and partition are the same idea expressed two ways.

7 Functions

A function from A to B is the simplest kind of relation: to each input there corresponds exactly one output. Most of the apparatus of political economy is built out of functions: utility functions, payoff functions, voting rules, demand functions, social choice functions. We unpack the definition carefully because subtle issues (does the codomain matter? what is the difference between range and codomain?) come up later.

Definition 26. A *function* from A to B, written $f : A \to B$, is a relation $f \subseteq A \times B$ such that for every $a \in A$ there is exactly one $b \in B$ with $(a, b) \in f$. We write $f(a) = b$ and call b the *value of f at a*. The set A is the *domain* and the set B is the *codomain*.[3]

[3] Two points worth flagging. First, the modern definition treats a function as a set of pairs satisfying the at-most-one-output condition. The older view treated a function as a *rule* for computing outputs from inputs — but two different rules can produce the same set of pairs (e.g., $x \mapsto x$ and $x \mapsto x + 0$), so the rule view does not pin down the function uniquely. The graph view does. Second, the codomain is part of the data of a function: $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = x^2$ is a different function from $g : \mathbb{Z} \to \mathbb{N}$ given by $g(x) = x^2$, even though they pair the same inputs with the same outputs. The difference matters for surjectivity (defined below) and for function composition.

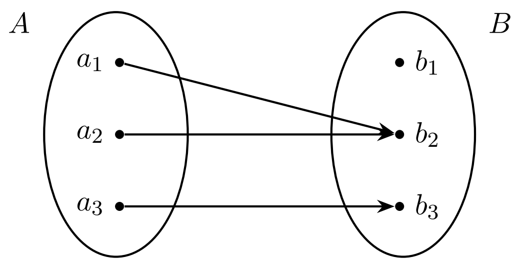
Definition 27. For a function $f : A \to B$:

- the *image* of a subset $S \subseteq A$ is $f(S) = \{f(a) : a \in S\}$;
- the *image* of f (also called the *range*) is $f(A) \subseteq B$;
- the *preimage* of a subset $T \subseteq B$ is $f^{-1}(T) = \{a \in A : f(a) \in T\}$.

The notation $f^{-1}(T)$ is for the preimage of a *set*; it does not require f to have an inverse.

Definition 28. A function $f : A \to B$ is:

- *injective* (or *one-to-one*) if $f(a) = f(a')$ implies $a = a'$;
- *surjective* (or *onto*) if $f(A) = B$, i.e., every $b \in B$ has at least one $a \in A$ with $f(a) = b$;
- *bijective* (or a *bijection*) if it is both injective and surjective.



The diagram above shows a function that is neither injective (a_1 and a_2 both map to b_2) nor surjective (b_1 has no preimage). Most functions encountered in the wild are like this: just functions, with no further structure.

Definition 29. For functions $f : A \to B$ and $g : B \to C$, the *composition* $g \circ f : A \to C$ is defined by $(g \circ f)(a) = g(f(a))$. The *identity function* on A is $\mathrm{id}_A : A \to A$ given by $\mathrm{id}_A(a) = a$.

Proposition 30. *A function $f : A \to B$ has an* inverse *$g : B \to A$ — that is, $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$ — if and only if f is bijective. The inverse, when it exists, is unique and is denoted f^{-1}.*

Proof. ($\Leftarrow$) If f is bijective, define $g(b)$ to be the unique $a \in A$ with $f(a) = b$ — existence by surjectivity, uniqueness by injectivity. Direct verification gives $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$. ($\Rightarrow$) Conversely, $g \circ f = \mathrm{id}_A$ forces f injective: if $f(a) = f(a')$, applying g to both sides gives $a = a'$. And $f \circ g = \mathrm{id}_B$ forces f surjective: every b equals $f(g(b))$. □

Example 31 (Utility function)**.** A utility function $u : X \to \mathbb{R}$ assigns a real number to each alternative in a set X. It is rarely injective and almost never surjective; we still call it a function. The pairs $(x, u(x))$ are what matters.

Example 32 (Voting rule)**.** Let X be a set of alternatives and let $L(X)$ denote the set of strict linear orders on X. A *voting rule* for n voters is a function $L(X)^n \to X$, mapping each profile of voter rankings to a chosen alternative. (If ties are allowed in the output, the codomain is $\mathcal{P}(X) \setminus \{\emptyset\}$ instead.)

8 The limits of "naive"

We have leaned freely on the principle that, given any property P, we can form the set of all things satisfying P. The principle is wrong.

Example 33 (Russell's paradox)**.** Suppose, for the sake of argument, we could form the set

$$R = \{x : x \notin x\},$$

the set of all sets that do not contain themselves as elements. Is R an element of R?

If $R \in R$, then R satisfies the defining property, so $R \notin R$ — contradiction. If $R \notin R$, then R satisfies the defining property, so $R \in R$ — contradiction. Either branch produces a contradiction, so the assumption that R is a set is wrong.[4]

The moral, for applied work: do not be tempted to write things like "the set of all sets" or "the universe of all mathematical objects." Always work relative to some already-existing set — subsets of $\mathbb{R}$, subsets of a strategy space, subsets of $\mathcal{P}(N)$ for some legislator set N. The places where this discipline matters are exactly the places where political economy actually operates, so the discipline costs nothing.

9 What's next

Three handouts pick up directly from this one:

- *Order theory.* Reflexive-antisymmetric-transitive relations are partial orders; adding a comparability requirement gives a total order. Lattices, suprema and infima, and the apparatus for monotone reasoning sit on top of these definitions.
- *Cardinality and infinity.* Two sets are equinumerous if there is a bijection between them. The bijection definition is what lets us compare the sizes of infinite sets, and it produces some genuinely surprising results — the rationals are equinumerous with the integers, but the reals are not.
- *First-order logic.* The formal language in which the axioms of set theory (ZFC) are stated. This handout's footnotes about the axiom of separation, the cumulative hierarchy, and proper classes can all be made fully precise once first-order logic is in hand.

For a fuller treatment at this level, the canonical reference is Halmos (1960).

[4] The repair, due to Zermelo and now standard, is to restrict the comprehension principle: instead of forming $\{x : P(x)\}$ for arbitrary P, you may only form $\{x \in A : P(x)\}$, where A is an already-existing set. This is the *axiom of separation* (sometimes *specification* or *subset* axiom). It blocks the Russell paradox: "the set of all sets that don't contain themselves" presupposes a universe of all sets, and that universe is itself not a set in axiomatic set theory — it is a *proper class*, an aggregate too large to be a set. The intuition behind the standard axiomatization (Zermelo–Fraenkel set theory, ZF, or with the axiom of choice ZFC) is the *cumulative hierarchy*: sets are built up in stages, each stage containing all subsets of the union of previous stages. There is no stage at which "the set of all sets" is formed, because no stage contains itself. The cumulative hierarchy is what makes the naive picture safe in practice: every set you actually construct in applied work lives at some stage and is unproblematic. The axiomatic apparatus is for set theorists who want to study sets themselves; for the rest of us, naive comprehension restricted to "x in some already-existing set" is enough. The formal language in which the axioms are stated is first-order logic — see the corresponding handout.

10 Exercises

Exercise 34. Show that $A \cap B = A$ if and only if $A \subseteq B$.

Exercise 35. For each of the following claims, prove it or give a counterexample:

1. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
2. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.
3. $A \times (B \cup C) = (A \times B) \cup (A \times C)$.
4. $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Exercise 36. Let $N = \{1, 2, \ldots, n\}$ be a set of n legislators, and let $\mathcal{P}(N)$ denote the set of possible coalitions. How many coalitions are there? How many *collections of coalitions* are there (i.e., $|\mathcal{P}(\mathcal{P}(N))|$)? List all coalitions of a three-member legislature: $\mathcal{P}(\{1, 2, 3\})$.

Exercise 37. Let $V = \{v_1, \ldots, v_n\}$ be a finite set of voters and $C = \{c_1, \ldots, c_m\}$ a finite set of candidates. A *preference profile* assigns to each voter a strict ranking of the candidates. Express the set of all preference profiles in terms of Cartesian products of suitable sets. How many preference profiles are there?

Exercise 38. Let V_1 and V_2 be the populations of two legislative districts (allow that some voters might be registered in both — exotic, but the construction handles it). Show that $|V_1 \sqcup V_2| = |V_1| + |V_2|$ counts each voter once for each district they're registered in, while $|V_1 \cup V_2| = |V_1| + |V_2| - |V_1 \cap V_2|$ (*inclusion–exclusion*) counts each voter exactly once. Conclude that $|V_1 \sqcup V_2| \geq |V_1 \cup V_2|$, with equality if and only if the districts share no voters.

Exercise 39. For each of the following relations on $\mathbb{Z}$, state which of {reflexive, symmetric, transitive, antisymmetric} hold, and identify which are equivalence relations.

1. $a \, R \, b$ iff $a = b$.
2. $a \, R \, b$ iff a and b have the same number of digits.
3. $a \, R \, b$ iff $|a - b| \leq 1$.
4. $a \, R \, b$ iff $a \cdot b > 0$.
5. $a \, R \, b$ iff a and b are both even, or both odd.

Exercise 40. Let V be a population of voters and define $v \sim v'$ if v and v' are registered in the same legislative district. Verify $\sim$ is an equivalence relation. Describe the equivalence classes. If $|V| = N$ and there are D districts, what is $\sum_v |[v]_\sim|/|V|$? (Be careful: each v contributes the size of its own class.)

Exercise 41. A symmetric quadratic loss function $L : \mathbb{R} \to \mathbb{R}$ is given by $L(x) = x^2$ (for instance, the loss to a candidate from positioning at policy x when the median voter's ideal point is 0). Compute:

1. $L([-2, 2])$ — the set of losses incurred from positions in the interval $[-2, 2]$.

2. $L^{-1}([1,4])$ — the set of positions producing losses in $[1,4]$.
3. $L^{-1}(L([-2,2]))$. (This is generally not equal to $[-2,2]$.)
4. $L(L^{-1}([-1,4]))$. (This is generally not equal to $[-1,4]$.)

For (3) and (4), explain in a sentence what failed and what it has to do with the function being non-injective and not surjective onto $[-1,4]$.

Exercise 42. Show that the composition of two injections is an injection, and that the composition of two surjections is a surjection. Conclude that the composition of two bijections is a bijection.

Exercise 43. Let $u : X \to \mathbb{R}$ be a utility function on a set of alternatives X. Define $x \sim y$ if $u(x) = u(y)$. Show that $\sim$ is an equivalence relation. Its equivalence classes are called *indifference classes*: explain why this is appropriate language.

Exercise 44. Explain in your own words why "the set of all sets" cannot be a set. (Hint: assume it is, call it S. Note that $S \in S$. Now consider $T = \{x \in S : x \notin x\}$ — a legitimate use of the axiom of separation since S is, by assumption, a set. Ask whether $T \in T$.)

References

Halmos, Paul R. (1960). *Naive Set Theory*. Princeton: Van Nostrand.