Domain restrictions and the median voter

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1 Motivation

The previous handout's impossibility constellation — Arrow, Sen, Wilson, Gibbard–Satterthwaite — is deeply structural under universal domain. No aggregation rule satisfies a small set of intuitive axioms when it must operate on every conceivable preference profile. But political-economy applications rarely involve preferences uniformly distributed over the set of all possible orderings. Voters facing a one-dimensional policy choice (federal-vs.-confederal, conservative-vs.-liberal, more-vs.-less redistribution) typically have preferences that decline as one moves away from the voter's ideal point. Roll-call records of legislative voting reveal preferences that vary systematically with legislators' positions on a low-dimensional ideological scale (Poole and Rosenthal 1985 and the entire NOMINATE literature). Voters' preferences over redistribution programs typically vary monotonically with income, with the cross-over from "wanting more" to "wanting less" occurring at a particular income threshold for each policy. Each of these is a structural claim about the empirical shape of preferences — a domain restriction — and each of them, when invoked, lifts the impossibility theorems' bite.

Three structural insights organize this handout. First, the simplest positive result is for the binary-alternative case: when there are only two alternatives, May (1952)'s theorem characterizes majority rule as the unique aggregation rule satisfying anonymity, neutrality, and positive responsiveness, with no conflict with Arrow's other axioms (§2). Second, the most powerful one-dimensional positive result is Black (1948)'s median voter theorem: when preferences are single-peaked over a linearly-ordered alternative space, the median voter's ideal point is the Condorcet winner, majority rule is well-behaved, and Arrow's impossibility does not bite (§3). The Downsian convergence-to-the-median result (Downs 1957) is the most-cited corollary in political-economy modeling. Third, the multi-dimensional analog is generic-fragile: Plott (1967)'s symmetry condition delivers a multi-dimensional median voter only under non-generic structural symmetry of the voter ideal-point distribution, and McKelvey (1976, 1979)'s chaos theorem shows that without that symmetry, agenda manipulation in multi-dimensional pairwise voting can move the outcome anywhere in the alternative space (§5). *Single-crossing preferences* (§4) is a related-but-distinct domain restriction with its own empirical pedigree (taxation, redistribution) and its own median-voter result.

This handout closes the social-choice cluster (#30–#31). The structural through-line — from impossibility under universal domain (handout #30) to existence and well-behavior under domain restrictions (this handout) — mirrors the methodological move that organizes most of welfare economics: when a general theorem is unavailable, restrict the domain in ways that make a positive result available, and study the substantive structure of the restriction.

2 Majority rule on two alternatives: May's theorem

The cleanest possible setting for preference aggregation is the binary-alternative case — a referendum, a yes/no vote, an up-or-down on a single proposal. With only two alternatives in X, every preference is automatically a complete ranking (a voter either prefers a to b, or b to a, or is indifferent), and the structural problem of aggregating cycles cannot arise. May (1952) characterizes majority rule axiomatically in this setting, and the result is the cleanest positive theorem in social choice.

Throughout this section, $X = \{a, b\}$ and we treat the aggregation rule as a function $F : \mathcal{R}^n \to \{a, b, \text{tie}\}$ that selects a winner (or declares a tie).

Definition 1 (May's axioms)**.** A binary aggregation rule F satisfies:

- **Decisiveness (D):** for every profile, F selects a, b, or declares a tie (and never returns "undefined").
- **Anonymity (A):** the rule depends only on the number of voters preferring each alternative, not on which specific voters do (i.e., F is invariant under permutations of voter labels).
- **Neutrality (N):** the rule treats a and b symmetrically (i.e., F commutes with the alternative-relabeling that swaps a and b).
- **Positive responsiveness (PR):** if $F(\boldsymbol{R}) = a$ or $F(\boldsymbol{R}) = \text{tie}$, and the profile $\boldsymbol{R}'$ differs from $\boldsymbol{R}$ only in that one voter has changed her preference from $b \succ a$ to $a \succ b$, then $F(\boldsymbol{R}') = a$.

The substantive readings. (D) and (A) are minimal procedural-fairness conditions: the rule must produce an answer, and it must not depend on which voter is labeled #1 versus #2. (N) is a no-status-quo-bias condition: the rule must not implicitly favor one alternative over the other in a structural way. (PR) is a monotonicity condition: an additional voter switching to support a must not hurt a's chances.

Theorem 2 (May 1952)**.** *Let $X = \{a, b\}$. A binary aggregation rule F satisfies (D), (A), (N), and (PR) if and only if F is simple majority rule (i.e., $F(\boldsymbol{R}) = a$ if and only if more voters prefer a than b, $F(\boldsymbol{R}) = b$ if and only if more voters prefer b than a, and $F(\boldsymbol{R}) = \text{tie}$ if and only if equal numbers prefer each).*

Proof sketch. By (A), the rule depends only on the count (n_a, n_b) of voters preferring a versus b (with possibly some indifferent voters factored out). By (N), if the count is (n_a, n_b) and the rule selects a, then on the count (n_b, n_a) the rule must select b. By (PR), the rule's selection is monotone in $n_a - n_b$. Combining: the rule selects a iff $n_a > n_b$, b iff $n_b > n_a$, and ties iff $n_a = n_b$ — which is majority rule. □

The theorem is the structural reason majority rule is the canonical aggregation rule on binary-alternative settings. All four axioms are individually defensible, and together they pin down the rule uniquely. Where Arrow's theorem says "no rule satisfies all the nice axioms with three or more alternatives," May's theorem says "with two alternatives, exactly one rule satisfies the nice axioms." The structural shift between $|X| = 2$ and $|X| \geq 3$ is the source of the impossibility cluster: with three or more alternatives, IIA-style cross-pair conditions become substantive, and the impossibility kicks in.

Example 3 (Referendum on a constitutional amendment)**.** A referendum offers voters a binary choice: ratify the proposed amendment (alternative a) or retain the status quo (alternative b). May's theorem says that majority rule is the unique aggregation rule satisfying (D), (A), (N), (PR). The substantive reading is that, in a properly-designed binary referendum, majority rule is essentially forced by the procedural-fairness axioms; the question of whether to use majority rule is structurally settled. Where ratification rules deviate from majority — requiring supermajorities for constitutional amendments, for instance — the deviation must come from giving up (N), the no-status-quo-bias axiom: the supermajority requirement structurally favors the status quo.

3 Single-peaked preferences and Black's theorem

Most political-economy modeling does not stop at binary alternatives. A legislator deciding among many spending levels for a program; a voter choosing among multiple candidates positioned along a left-right scale; a constitutional convention selecting from many possible federal arrangements — in each case the alternative space is multi-element, and Arrow's impossibility starts to bite. The most powerful escape is via *single-peakedness*: a structural restriction on individual preferences that captures the empirical shape of preferences over one-dimensional policy spaces, and that delivers a clean positive result — the median voter's ideal point is the Condorcet winner, and majority rule is well-behaved.

The setup throughout this section is a one-dimensional alternative space, $X \subseteq \mathbb{R}$ (or any totally ordered set), so that we can speak of one alternative being "between" two others.

Definition 4 (Single-peaked preferences)**.** A voter i's preference $\succsim_i$ on a totally ordered X is *single-peaked* if there exists an alternative $x_i^* \in X$ (the voter's *ideal point* or *peak*) such that for any $a, b \in X$:

- if $a < b \leq x_i^*$ or $x_i^* \leq b < a$, then $b \succ_i a$;
- the preference is strictly decreasing in distance from x_i^* on each side (closer-to-x_i^* is strictly better than farther-from-x_i^* on the same side).

A profile $\boldsymbol{R} = (\succsim_1, \ldots, \succsim_n) \in \mathcal{R}^n$ is single-peaked if every $\succsim_i$ is single-peaked.

The intuition is the spatial-voting picture: each voter has an ideal point on a one-dimensional policy continuum, and her utility falls off symmetrically as one moves away. Quadratic-loss utility $u_i(x) = -(x - x_i^*)^2$ is the canonical example (and the prototypical case considered in the convex-functions handout #21 and the welfare handout #28). Single-peakedness is more general than quadratic loss: any utility function that is unimodal on the alternative space — with a single peak and monotone decline on either side — generates single-peaked preferences.

Example 5 (Single-peaked preferences with three voters)**.** Three voters with ideal points $x_1^* = 1$, $x_2^* = 2$, $x_3^* = 3$ on the policy space $X = \{1, 2, 3, 4, 5\}$, with preferences declining monotonically in distance from each voter's ideal point. Voter 1's preference order: $1 \succ_1 2 \succ_1 3 \succ_1 4 \succ_1 5$. Voter 2's: $2 \succ_2 (1, 3) \succ_2 (4, 5)$ where the parenthesized pairs may be ranked either way (say $2 \succ_2 1 \succ_2 3 \succ_2 4 \succ_2 5$ for definiteness). Voter 3's: $3 \succ_3 (2, 4) \succ_3 (1, 5)$ (say $3 \succ_3 4 \succ_3 2 \succ_3 5 \succ_3 1$). The profile is single-peaked because each voter's preferences decline as one moves away from her ideal point.

The headline positive result.

Theorem 6 (Black's median voter theorem, 1948)**.** *Let $X \subseteq \mathbb{R}$ be a totally ordered alternative space, and let $\boldsymbol{R} = (\succsim_1, \ldots, \succsim_n) \in \mathcal{R}^n$ be a single-peaked profile with an odd number of voters. Let x_m^* be the median of the voters' ideal points. Then x_m^* is the* Condorcet winner*: it majority-defeats every other alternative.*

Proof. Let $y \in X$ with $y \neq x_m^*$. Without loss of generality, assume $y < x_m^*$ (the symmetric case is identical). The set of voters i with $x_i^* \geq x_m^*$ has size at least $(n+1)/2$ (a majority, since x_m^* is the median and n is odd). For any such voter i, we have $y < x_m^* \leq x_i^*$, so x_m^* is strictly closer to x_i^* than y is. By single-peakedness, voter i strictly prefers x_m^* to y. So at least $(n+1)/2$ voters prefer x_m^* to y, a strict majority. Hence x_m^* majority-defeats y. □

The theorem has three immediate substantive corollaries.

Corollary 7 (No Condorcet cycles under single-peakedness)**.** *On the single-peaked domain with an odd number of voters, the pairwise-majority relation is transitive and has the median voter's ideal point as its top element. The Condorcet cycle is impossible on this domain.*

Corollary 8 (Arrow's impossibility relaxes on the single-peaked domain)**.** *The aggregation rule that returns the social ranking induced by pairwise majority voting is well-defined on the single-peaked domain (no cycles arise), satisfies (P), (IIA), and (D), and provides an aggregation rule consistent with all four of Arrow's axioms when (U) is restricted to single-peaked profiles.*

Corollary 9 (Median-voter convergence in two-candidate competition)**.** *In two-candidate Downsian competition with single-peaked voters, both candidates have an incentive to converge to the median voter's ideal point as their announced policy. (Downs, 1957)*

The Downsian corollary is the workhorse application: it is the structural foundation for the "median voter" framework that organizes electoral-competition theory, comparative politics, and large parts of formal political economy. The substantive claim is that under single-peaked voter preferences, the equilibrium of two-candidate spatial competition has both candidates announcing the median voter's ideal as their policy — both candidates converge.[1]

4 Single-crossing preferences

Single-peakedness imposes structure on each voter's preferences *in isolation*: each voter has an ideal point on a one-dimensional space and her preferences decline as one moves away. An alternative domain restriction — *single-crossing* — imposes structure on *how preferences vary across voters* rather than on each voter's preferences taken alone. The two restrictions are related but distinct, and single-crossing is the canonical structural input to political-economy models of redistribution and taxation, where the "voter ordering" is the income distribution and the relevant structural fact is that voter preferences over redistribution policies are monotone in income.

[1] The single-peaked-implies-no-Condorcet-cycle theorem is a special case of a more general line of work on *value-restricted* preferences. Sen (1966) and Inada (1969) characterize the broader class of profiles on which majority rule is transitive: a profile is value-restricted if for every triple of alternatives $\{a, b, c\}$, there is some alternative that no voter ranks "in the middle" (i.e., between the other two). Single-peakedness is one specific way of being value-restricted (the extreme alternative is never ranked in the middle), but other ways exist (single-troughed preferences, single-crossing preferences — the next section — and others). The general lesson is that domain restrictions which guarantee well-behaved aggregation are characterized by structural conditions on how preferences vary across voters, with single-peakedness as the most-cited but by no means unique example. Austen-Smith and Banks (1999, Ch. 10) works through the value-restriction framework.

Definition 10 (Single-crossing)**.** Let $X \subseteq \mathbb{R}$ be a totally-ordered alternative space and let voters be totally ordered (say by income, ability, or another structural attribute) so that we may write $i < j$ to mean "voter i comes before voter j." A profile $\boldsymbol{R}$ is *single-crossing* if for every pair of alternatives $a < b$ in X, there exists a threshold voter $i^*(a, b)$ such that:

- all voters $i \leq i^*(a, b)$ have $a \succsim_i b$ (or all have $a \precsim_i b$ — one direction or the other);
- all voters $i > i^*(a, b)$ have the opposite preference.

The intuition: as one moves along the voter ordering, voters' preferences over any pair of alternatives "cross over" at most once. Equivalently: the pairwise-comparison preference of each voter is a monotone function of the voter's position in the ordering.

Theorem 11 (Median voter under single-crossing)**.** *Suppose $\boldsymbol{R}$ is single-crossing on X with respect to a voter ordering. Then the alternative most preferred by the median voter (in the voter ordering) majority-defeats every other alternative.*

Proof sketch. By single-crossing, the median voter's preferences "split the voter set into two groups" for every pair of alternatives. The two groups together with the median voter form a majority on either side of the cross-over, depending on the preference. The median voter's most-preferred alternative is therefore in the majority on every pairwise comparison. □

The substantive content of the theorem is parallel to Black's: under single-crossing, the median voter's preferences are decisive, and majority rule is well-behaved on the single-crossing domain. The structural difference between Black's theorem and the single-crossing theorem is that Black's median is computed in the alternative space (the median ideal point on the policy line), while the single-crossing median is computed in the voter space (the median voter, by some attribute ordering). For political-economy applications where voters are naturally ordered by an attribute (income, age, ideological position), single-crossing is often the more natural structural assumption.

Example 12 (Redistribution and the median-income voter)**.** A classic political-economy application is the Meltzer and Richard (1981) model of redistribution. Voters are ordered by income y, and each voter has preferences over a one-dimensional redistribution policy (say, the income tax rate $\tau \in [0, 1]$). Voter i's preferred tax rate is decreasing in y_i (richer voters want less redistribution). The pairwise comparison of two tax rates $\tau_1 < \tau_2$ has all voters with income below some threshold preferring τ_2 (more redistribution) and all voters above the threshold preferring τ_1. The single-crossing structure is satisfied; the median-voter theorem applies; the equilibrium tax rate is the one preferred by the median-income voter. The substantive prediction: redistribution increases as the gap between the median and mean income widens (since the median-income voter benefits more from redistribution as her position in the income distribution is further below the mean). The model is the workhorse for formal political-economy of redistribution, with empirical-implication chains running from income-distribution shape to redistribution-policy outcomes.

5 Beyond one dimension: Plott and McKelvey

The median voter theorem of §3 is a one-dimensional result. In multi-dimensional alternative spaces, single-peakedness alone is not enough: the natural multi-dimensional analog requires an additional symmetry condition, and the symmetry is structurally fragile. Without it, the multi-dimensional voting equilibrium fails to exist, and a striking pathology emerges — agenda-controlled pairwise voting can move the eventual outcome anywhere in the multi-dimensional alternative space. The two relevant theorems are Plott (1967) on the symmetry condition and McKelvey (1976, 1979) on the chaos that follows from its failure.

The setup throughout this section: a multi-dimensional alternative space $X = \mathbb{R}^k$ with $k \geq 2$ (typically $k = 2$ for visualization), and voters with Euclidean (radially-symmetric) preferences, $u_i(x) = -\|x - x_i^*\|^2$. The relevant structural object is the configuration of voter ideal points $x_1^*, \ldots, x_n^*$ in $\mathbb{R}^k$.

Definition 13 (Plott's symmetry condition)**.** A configuration of voter ideal points $x_1^*, \ldots, x_n^* \in \mathbb{R}^k$ satisfies *Plott's symmetry condition* at $x_0 \in \mathbb{R}^k$ if for every voter i with $x_i^* \neq x_0$, there exists a paired voter j with $x_j^* = 2x_0 - x_i^*$ (the mirror-image of x_i^* through x_0).

The symmetry condition says that the voter configuration is *radially symmetric* around x_0 — for every voter on one side of x_0, there is a paired voter equidistant on the opposite side.

Theorem 14 (Plott 1967)**.** *Suppose voters have Euclidean preferences and the configuration of ideal points satisfies Plott's symmetry condition at some $x_0 \in \mathbb{R}^k$. Then x_0 is a Condorcet winner: it majority-defeats every other alternative under pairwise voting.*

Proof sketch. For any $y \neq x_0$, partition the voters into pairs (i, j) where $x_j^* = 2x_0 - x_i^*$. For each such pair, exactly one of the two voters strictly prefers x_0 to y, and the other strictly prefers y to x_0 — but the pair's contribution to the majority count cancels except at the boundary cases. Voters whose ideal point coincides with x_0 all prefer x_0 to y. The argument shows x_0 ties or weakly defeats y; under generic conditions on y, x_0 strictly defeats y. □

The substantive observation is that Plott's symmetry is a non-generic condition: small perturbations of voter ideal points generically destroy the symmetry, and when the symmetry fails the Condorcet winner ceases to exist.

The chaos that follows from the failure is striking.

Theorem 15 (McKelvey 1976, McKelvey 1979 chaos theorem)**.** *Suppose voters have Euclidean preferences in $\mathbb{R}^k$ with $k \geq 2$, and the configuration of ideal points fails Plott's symmetry condition. Then for any pair of alternatives $x_0, x^* \in \mathbb{R}^k$, there exists a finite sequence of alternatives $x_0, x_1, x_2, \ldots, x_T = x^*$ such that each x_{t+1} majority-defeats x_t in pairwise comparison.*

The substantive reading is dramatic: in multi-dimensional voting without Plott's symmetry, pairwise majority rule is so unstable that an agenda-setter who controls the sequence of pairwise votes can move the eventual outcome anywhere in the alternative space, starting from any initial alternative. The chaos theorem says that the multi-dimensional median voter does not exist generically, and that without institutional structure, multi-dimensional voting has no equilibrium.

The political-economy implications are profound and have organized a substantial literature. Two strands of response.[2]

[2]The McKelvey chaos theorem is the structural foundation for two distinct lines of political-economy work that aim to restore equilibrium in multi-dimensional voting. The first is Shepsle (1979) and the *structure-induced equilibrium* (SIE) framework: institutional structures — committee jurisdictions, sequential voting on issue-by-issue agendas, supermajority requirements on certain dimensions — can effectively decompose a multi-dimensional alternative space

Structure-induced equilibrium (Shepsle, 1979): institutional structures — committee jurisdictions, agenda control, sequential voting on issue-by-issue agendas — can decompose multi-dimensional voting into a sequence of one-dimensional sub-decisions, on each of which Black's median voter theorem applies. The substantive insight is that the apparent stability of legislative policy outcomes is an artifact not of structural equilibrium but of *institutional decomposition*, where committee jurisdictions and agenda procedures effectively force a one-dimensional structure onto each sub-decision.

Heresthetic (Riker, 1986): skilled political actors exploit the McKelvey-chaos vulnerability by manipulating the dimensional structure of an issue. A politician facing a losing position on the existing agenda can introduce a new dimension that re-shuffles the coalitional structure — the structural pathology of the chaos theorem becomes an instrument for political maneuvering rather than a curse. The substantive lesson is that political institutions and political actors are not passively subject to McKelvey chaos; they actively exploit and constrain it.

6 What's next

This handout closes the social-choice cluster (#30–#31). Three strands extend it.

Mechanism design (forthcoming game-theory cluster) takes up the question of strategic implementation under domain restrictions and information asymmetries. The Gibbard–Satterthwaite impossibility of #30 forces strategy-proof implementation to be dictatorial under universal domain; under restricted domains (e.g., quasi-linear preferences in private-values environments), Vickrey–Clarke–Groves mechanisms restore strategy-proof implementation, and Myerson's optimal-auction theory characterizes Bayesian incentive-compatible mechanisms. The line connects the impossibility of #30, the domain restrictions of this handout, and the equilibrium concepts of the forthcoming game-theory cluster.

Strategic voting and pivotal-voter models (forthcoming game-theory cluster) take up the substantive question of how rational voters behave in elections under strategic conditions, with the median voter theorem as the limiting reference case but realistic models incorporating costs of voting, uncertainty about other voters, and informational asymmetries. The pivotal-voter model of Ledyard (1984) and the rational-voter framework of Palfrey and Rosenthal (1985) are the standard references.

Empirical scaling and ideal-point estimation are the empirical implementation of single-peakedness assumptions on roll-call data. The NOMINATE methodology (Poole and Rosenthal, 1985, 1997) fits a low-dimensional spatial model to roll-call records of legislators' votes, recovering each legislator's ideal point on the inferred ideological dimensions. Bayesian variants (the IDEAL methodology of Clinton, Jackman, and Rivers (2004)) provide posterior distributions over ideal points and other parameters. The methodology is the empirical workhorse of the legislative-studies literature and rests

into one-dimensional sub-decisions, on each of which Black's median voter theorem applies. SIE is the structural answer to "why we observe stable policy outcomes in legislatures despite the chaos theorem"; the answer is that legislative institutions do not implement pure pairwise multi-dimensional majority rule but rather a structured version that decomposes the problem. The second is Riker (1986)'s concept of *heresthetic* — the political art of manipulating the dimensional structure of an issue to one's advantage. A skilled heresthetician changes the agenda or reframes the issue to rearrange the structural geometry of the voting problem, exploiting the McKelvey-chaos vulnerability of the existing decision structure. Both lines of work treat McKelvey not as an impossibility to be wished away but as a constraint to be navigated by institutional design or strategic political maneuvering. Austen-Smith and Banks (1999, 2005) works through both lines in detail with the political-economy applications as the centerpiece.

on the structural assumption that legislators' preferences are single-peaked over a low-dimensional ideological space — exactly the structural condition Black's theorem operates on.

For graduate-level treatments at this handout's level: Austen-Smith and Banks (1999, 2005) treats both the impossibility and the positive theory in detail with the political-science applications throughout; Black (1948, 1958) are the foundational references on single-peakedness and the median voter; McKelvey (1976, 1979) are the canonical references on the chaos theorem; Shepsle (1979) on structure-induced equilibrium; Persson and Tabellini (2000) for the macroeconomic-political-economy applications.

7 Exercises

Exercise 16. *May's axioms applied.* Consider the following binary aggregation rules. For each, determine which of (D), (A), (N), (PR) it satisfies and which it violates. (a) Simple majority rule. (b) Two-thirds supermajority for a over the status quo b. (c) Status-quo bias: b wins ties. (d) Random-dictator: a randomly chosen voter's preference is the rule's selection.

Exercise 17. *Single-peakedness verification.* For each of the following utility functions on $X = [0, 10]$, verify whether the implied preferences are single-peaked. (a) $u(x) = -(x-4)^2$. (b) $u(x) = -|x-4|$. (c) $u(x) = (x-4)^2$ (note the sign). (d) $u(x) = -(x-2)^2 - (x-8)^2$ (a two-peaked utility). (e) $u(x) = e^{-(x-4)^2}$ (Gaussian).

Exercise 18. *Black's theorem applied.* Five voters with ideal points $x_1^* = 1, x_2^* = 3, x_3^* = 5, x_4^* = 7, x_5^* = 9$ on $X = \{1, 2, \ldots, 10\}$, with quadratic-loss preferences. (a) Identify the median voter's ideal point. (b) Show that this ideal point majority-defeats $x = 6$ in pairwise comparison. (c) Show it majority-defeats $x = 4$. (d) Conclude that it is the Condorcet winner.

Exercise 19. *Downsian convergence.* Two candidates A and B choose policy positions $x_A, x_B \in [0, 1]$. Voters are uniformly distributed on $[0, 1]$ with quadratic-loss preferences (each voter votes for the candidate closer to her ideal point, splitting ties). Each candidate seeks to maximize her vote share. (a) Find the Nash equilibrium of the two-candidate competition. (b) Show that both candidates choose the median voter's ideal point ($x_A = x_B = 1/2$). (c) Discuss in two sentences how the result depends on the single-peaked-quadratic-loss assumption (and what fails if voters have, e.g., abstention costs).

Exercise 20. *When single-peakedness fails.* Construct a profile of three voters and three alternatives in which preferences are NOT single-peaked over the natural ordering of alternatives, and verify that the resulting pairwise majority comparisons exhibit a Condorcet cycle. Compare with Example 5 (which is single-peaked and has no cycle).

Exercise 21. *Single-crossing verification.* Three voters with the following preferences over $X = \{a, b, c\}$ (with $a < b < c$):

- Voter 1: $a \succ_1 b \succ_1 c$
- Voter 2: $b \succ_2 a \succ_2 c$
- Voter 3: $b \succ_3 c \succ_3 a$

(a) For each pair of alternatives, identify which voters prefer the smaller to the larger. (b) Verify that the profile is single-crossing with respect to the voter ordering $1 < 2 < 3$. (c) Identify the median voter and her most-preferred alternative.

Exercise 22. *Meltzer–Richard redistribution.* A continuum of voters has incomes $y \in [0, 10]$ uniformly distributed; income is taxed at rate $\tau \in [0, 1]$ and the proceeds are redistributed equally. Each voter's after-redistribution income is $y(1-\tau) + \tau\bar{y}$, where $\bar{y} = 5$ is the mean income. (a) Argue that voter y's preferred tax rate is $\tau^*(y) = 1$ if $y < \bar{y}$, $\tau^*(y) = 0$ if $y > \bar{y}$. (b) Verify single-crossing. (c) Find the median-voter tax rate. (d) Discuss in two sentences how the result generalizes when income distribution is right-skewed (median below mean).

Exercise 23. *Plott symmetry and its failure.* Three voters in $\mathbb{R}^2$ with ideal points $x_1^* = (0, 0)$, $x_2^* = (2, 0)$, $x_3^* = (1, 1)$. (a) Argue that the ideal-point configuration does NOT satisfy Plott's symmetry condition at any candidate equilibrium. (b) Conclude (per McKelvey's theorem) that pairwise majority rule has no Condorcet winner. (c) Now perturb to $x_1^* = (0, 0)$, $x_2^* = (2, 0)$, $x_3^* = (1, 0)$ (collinear voters). Does Plott's symmetry hold? Identify the equilibrium if so.

Exercise 24. *McKelvey chaos in action.* Consider a three-voter setting in $\mathbb{R}^2$ with ideal points as in Exercise 23(a). Choose a starting alternative x_0 and a target alternative x^* (any two distinct points in $\mathbb{R}^2$). Without computing explicit pairwise winning sets, sketch in two or three sentences the argument that there is a sequence of pairwise majority comparisons taking x_0 to x^* (drawing on the McKelvey chaos theorem).

Exercise 25. *Agenda-setter exploitation.* A three-voter committee in $\mathbb{R}^2$ has ideal points failing Plott's symmetry condition. The committee chair (the agenda-setter) is empowered to choose the sequence of pairwise comparisons. (a) Argue, using the McKelvey chaos theorem, that the chair can choose the agenda so as to implement any final policy in $\mathbb{R}^2$ that she prefers. (b) Discuss in two or three sentences the political-economy implication: the McKelvey chaos theorem makes agenda-control institutionally consequential. Connect to Shepsle (1979)'s structure-induced equilibrium framework as the institutional response to chaos.

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