Continuity

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1 Motivation

Continuity is the formal version of "a small change in the input produces a small change in the output." For political economists this is what underwrites comparative statics (a small change in a parameter shifts the equilibrium by a small amount), what makes best-response correspondences amenable to fixed-point arguments (Brouwer's theorem requires continuity), and what justifies the use of calculus on objective functions (a continuous function on a compact set has a maximum, by the extreme value theorem). Most of the heavy-lift theorems in PE depend on continuity somewhere, and the discontinuities — threshold effects in collective action, regime changes, tipping points, ties in plurality voting — are exactly the places where the standard machinery breaks and the analysis gets harder.

This handout is the payoff for the previous two. Sequences (and convergence in particular) gave us one characterization of continuity: f is continuous at a point iff it preserves limits of sequences converging to that point. Open and closed sets gave us the topological vocabulary in which the global theorems live. We will work through the ϵ-δ definition and its sequential and topological reformulations, then prove the two flagship theorems — the intermediate value theorem (continuity on a connected set takes all intermediate values) and the extreme value theorem (continuity on a compact set attains a maximum and a minimum) — and close with uniform continuity, where compactness automatically upgrades pointwise continuity to a stronger global form.

2 Continuity

What does it mean to say a model is *robust* — that small perturbations to inputs lead to only small changes in outputs? In political-economy modeling this question is everywhere: a small shift in the median voter's ideal point should produce only a small shift in equilibrium policy; a marginal change in a candidate's platform should not flip an election outcome by itself; a one-percentage-point change in a tax rate should not detonate the equilibrium of a market. When we believe these things we are silently assuming that the relevant objects are continuous functions of the relevant inputs, and continuity is the formal capture of that assumption.

The ϵ-δ formulation is the workhorse. It is built by analogy with the ϵ-N definition of a limit: "small change in input $\Rightarrow$ small change in output," with the input proximity controlled by a δ that may depend on ϵ and on the point.

Definition 1. Let $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$, and let $\boldsymbol{x} \in A$. We say f is *continuous at $\boldsymbol{x}$* if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\boldsymbol{y} \in A \text{ and } d(\boldsymbol{y}, \boldsymbol{x}) < \delta \implies d(f(\boldsymbol{y}), f(\boldsymbol{x})) < \epsilon.$$

We say f is *continuous on A* (or just *continuous*, when A is understood) if f is continuous at every $\boldsymbol{x} \in A$.

Three things to notice. First, the metric d on the input side may live in $\mathbb{R}^n$ and the metric d on the output side in $\mathbb{R}^m$ — the same letter does double duty. Second, δ is allowed to depend on both ϵ *and* $\boldsymbol{x}$ — the proximity required to keep f within ϵ may need to be tighter near some points than near others. (Promoting this dependence to a uniform one is the work of §5.) Third, the definition only constrains f near points of A: if a point $\boldsymbol{x} \in A$ is isolated — has an open ball containing no other point of A — then continuity at $\boldsymbol{x}$ is automatic.

The sequential characterization, promised at the end of the previous handout, is the practical tool.

Theorem 2 (Sequential characterization of continuity)**.** *Let $f : A \to \mathbb{R}^m$ and $\boldsymbol{x} \in A$. Then f is continuous at $\boldsymbol{x}$ if and only if for every sequence $(\boldsymbol{x}_k)$ in A with $\boldsymbol{x}_k \to \boldsymbol{x}$, we have $f(\boldsymbol{x}_k) \to f(\boldsymbol{x})$.*

Proof. ($\Rightarrow$) Assume f continuous at $\boldsymbol{x}$, and let $\boldsymbol{x}_k \to \boldsymbol{x}$. Given $\epsilon > 0$, choose $\delta > 0$ as in Definition 1, then choose K such that $d(\boldsymbol{x}_k, \boldsymbol{x}) < \delta$ for $k \geq K$. Then $d(f(\boldsymbol{x}_k), f(\boldsymbol{x})) < \epsilon$ for $k \geq K$, so $f(\boldsymbol{x}_k) \to f(\boldsymbol{x})$.

($\Leftarrow$, contrapositive) Assume f is not continuous at $\boldsymbol{x}$. Then for some $\epsilon_0 > 0$ and every $\delta > 0$, there exists $\boldsymbol{y} \in A$ with $d(\boldsymbol{y}, \boldsymbol{x}) < \delta$ but $d(f(\boldsymbol{y}), f(\boldsymbol{x})) \geq \epsilon_0$. Take $\delta = 1/k$ to produce $\boldsymbol{x}_k \in A$ with $d(\boldsymbol{x}_k, \boldsymbol{x}) < 1/k$ and $d(f(\boldsymbol{x}_k), f(\boldsymbol{x})) \geq \epsilon_0$. Then $\boldsymbol{x}_k \to \boldsymbol{x}$ but $f(\boldsymbol{x}_k) \not\to f(\boldsymbol{x})$. □

There is a cleaner topological reformulation as well, which makes continuity a statement about open sets only — with no ϵ, δ, or even metric in sight.[1] We will not lean on the topological definition formally below, but it is in the background of essentially every proof.

The basic algebra of continuity is what one would hope.

Proposition 3. *Let $f, g : A \to \mathbb{R}$ be continuous at $\boldsymbol{x} \in A$, and let $\alpha \in \mathbb{R}$. Then αf, $f + g$, fg, and (if $g(\boldsymbol{x}) \neq 0$) f/g are continuous at $\boldsymbol{x}$. The composition $g \circ f$ of continuous functions is continuous at every point where it is defined.*

We omit the proofs — they follow either by direct ϵ-δ chasing (mimicking the algebra-of-limits proof from the sequences handout) or by sequential characterization (let $\boldsymbol{x}_k \to \boldsymbol{x}$ and apply Proposition 3 of the sequences handout). The composition rule is what makes "continuity" a robust notion: built-up combinations of continuous building blocks (polynomials, exponentials, trig functions, log) are continuous wherever they are defined.

Example 4 (Continuous building blocks)**.** Constants and the identity $f(x) = x$ are continuous on $\mathbb{R}$. By the algebra of continuity, every polynomial is continuous on $\mathbb{R}$, every rational function is continuous on its domain (the complement of the zero set of the denominator), and every composition of these with the standard transcendental functions is continuous wherever the composition makes sense.

[1] The topological definition is that $f : X \to Y$ between topological spaces is continuous iff $f^{-1}(V)$ is open in X for every open $V \subseteq Y$. (Equivalently, $f^{-1}(F)$ is closed for every closed F.) The equivalence with the ϵ-δ definition in $\mathbb{R}^n$ is direct: open sets are exactly unions of open balls, and the ϵ-δ statement says exactly that the preimage of every open ball is open. The topological version is the right one for several reasons: it is shorter, it makes the proofs of the basic theorems (composition of continuous functions is continuous; image of compact is compact; image of connected is connected) almost trivial, and it generalizes immediately to settings where there is no obvious distance function. The ϵ-δ formulation is what one actually uses for hands-on calculations in $\mathbb{R}^n$ — chasing the dependence of δ on ϵ and on the point is what proves a specific function continuous — so both forms have their uses.

Example 5 (A discontinuous function from a tie-breaking rule)**.** Suppose two candidates A and B have ideal points 0 and 1 on the real line, and a voter at v votes for A iff $v < 1/2$ (with the convention that the voter at exactly 1/2 goes to A). Let $f(v)$ be the indicator that the voter votes for A: $f(v) = 1$ for $v \leq 1/2$ and $f(v) = 0$ for $v > 1/2$. Then f is continuous at every $v \neq 1/2$, but discontinuous at $v = 1/2$: $f(1/2) = 1$, but $f(1/2 + 1/k) = 0$ for every k, so $f(1/2 + 1/k) \to 0 \neq 1$. Discontinuities of this sort — driven by tie-breaking rules, threshold rules, regime changes — are pervasive in political-economy models, and they are exactly where the basic existence theorems become subtler (and where pure-strategy equilibria can fail to exist, motivating mixed strategies).

3 The intermediate value theorem

Suppose excess demand for some good is positive when the price is low (everyone wants to buy) and negative when the price is high (everyone wants to sell). Common sense says that, somewhere in between, there is a price at which the market clears. The intermediate value theorem is the formal version of this common sense, and it is the topological backbone of essentially every market-clearing existence argument in economics. The hypothesis we cash in is connectedness, from the previous handout; the punchline is that continuous images of connected sets do not skip values.

The $\mathbb{R}$-specific version is what one usually invokes: a continuous function on $[a, b]$ that takes both a positive and a negative value somewhere has a zero somewhere.

Theorem 6 (Intermediate value theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. For every y between $f(a)$ and $f(b)$, there exists $c \in [a, b]$ with $f(c) = y$.*

Proof. By replacing f with $f - y$ if necessary, it suffices to prove the case $y = 0$ and $f(a) < 0 < f(b)$ (or the reverse, which is symmetric). Let $S = \{x \in [a, b] : f(x) \leq 0\}$. The set S is nonempty ($a \in S$) and bounded above (by b), so by the LUB property has a supremum $c \in [a, b]$.

We claim $f(c) = 0$. Suppose $f(c) > 0$. By continuity, there is $\delta > 0$ such that $f(x) > 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. But then no $x \in (c - \delta, c]$ is in S, so $c - \delta$ is also an upper bound for S, contradicting that c is the least one. Suppose $f(c) < 0$. By continuity, there is $\delta > 0$ such that $f(x) < 0$ for all $x \in (c - \delta, c + \delta) \cap [a, b]$. But then $\min(c + \delta/2, b) \in S$ and exceeds c, contradicting that c is an upper bound. So $f(c) = 0$. □

The proof leans directly on the LUB property of $\mathbb{R}$, which is the order-theoretic content of connectedness. (The general topological version of the IVT — the continuous image of a connected set is connected, plus the $\mathbb{R}$-connected-iff-interval theorem from the previous handout — gives the same conclusion in a one-line argument; we use the LUB version here because it is the most commonly seen.)

Example 7 (Existence of a market-clearing price)**.** Let $z : [0, P] \to \mathbb{R}$ be a continuous excess-demand function: $z(p)$ is the difference between aggregate demand and aggregate supply when the price is p. Suppose $z(0) > 0$ (excess demand at zero price) and $z(P) < 0$ (excess supply at a sufficiently high price). The IVT guarantees a market-clearing price $p^* \in (0, P)$ with $z(p^*) = 0$. The continuity of z is doing the work — if supply or demand jumps discontinuously (e.g., minimum-bid thresholds, capacity constraints with sharp cutoffs), the IVT can fail and there may be no market-clearing price at all.

4 The extreme value theorem

Why does the consumer's utility-maximization problem have a solution? In the standard story: the utility function is continuous, the budget set is bounded (income limits the bundle) and closed (the constraints are weak inequalities), and "closed and bounded" is enough to guarantee that the maximum exists. The same reasoning underwrites firm profit maximization, social-planner problems, and (with a small additional convexity hypothesis) the existence proofs for Nash equilibrium in game theory. The mathematical content of all of these is one theorem: a continuous function on a compact set attains its maximum and its minimum. It is the most-invoked existence result in mathematical economics, and it is the reason the standard proof recipe begins "let K be compact, let f be continuous, then…"

Theorem 8 (Extreme value theorem)**.** *Let $K \subseteq \mathbb{R}^n$ be compact and let $f : K \to \mathbb{R}$ be continuous. Then f attains its supremum and its infimum on K: there exist $\boldsymbol{x}^*, \boldsymbol{x}_* \in K$ with*

$$f(\boldsymbol{x}^*) = \sup_{\boldsymbol{x}\in K} f(\boldsymbol{x}), \qquad f(\boldsymbol{x}_*) = \inf_{\boldsymbol{x}\in K} f(\boldsymbol{x}).$$

Proof. We prove the supremum case; the infimum case is symmetric. Let $M = \sup_{\boldsymbol{x}\in K} f(\boldsymbol{x}) \in \mathbb{R} \cup \{+\infty\}$. By the definition of supremum, there is a sequence $(\boldsymbol{x}_k)$ in K with $f(\boldsymbol{x}_k) \to M$ (taking $M = +\infty$ to mean unbounded). By the sequential characterization of compactness (Theorem 25 of the previous handout), $(\boldsymbol{x}_k)$ has a subsequence $\boldsymbol{x}_{k_j} \to \boldsymbol{x}^* \in K$. By the sequential characterization of continuity, $f(\boldsymbol{x}_{k_j}) \to f(\boldsymbol{x}^*)$. But $f(\boldsymbol{x}_{k_j})$ is a subsequence of $f(\boldsymbol{x}_k) \to M$, so it also converges to M. By uniqueness of limits, $M = f(\boldsymbol{x}^*) \in \mathbb{R}$, in particular finite, and the supremum is attained at $\boldsymbol{x}^*$. □

The proof is just two lines once the sequential characterizations of compactness and continuity are in place. The substance is in the prior handouts.

Example 9 (Existence of an optimum)**.** A continuous utility function $u : K \to \mathbb{R}$ on a compact feasible set K has a maximizer in K. This is the abstract content of the standard "the consumer's optimization problem has a solution" opening line in microeconomic theory: the budget set, defined by linear constraints intersected with the non-negative orthant and bounded by income, is closed and bounded, hence compact; the utility function is assumed continuous; EVT gives the existence of an optimal bundle. The same template applies to firm-side optimization, social-planner problems, and existence proofs for fixed-point-based equilibria.

Both compactness and continuity are needed — removing either one breaks the conclusion. The function $f(x) = x$ on the closed but unbounded $[0, \infty)$ has no maximum (closed but not compact). The function $f(x) = 1/x$ on the bounded but not closed $(0, 1]$ has no maximum on its domain (the supremum is $+\infty$). And the discontinuous function $f(x) = x$ for $x \in [0, 1)$, $f(1) = 0$ on the compact $[0, 1]$ has no maximum (the supremum 1 is not attained).

5 Uniform continuity

Sometimes pointwise continuity is not quite the right hypothesis. A statement like "this estimator is consistent uniformly across the parameter space," or "this approximation has error at most ϵ at every point," or "this comparative-statics result has bounded error throughout the relevant region" is asking for the same δ to work everywhere — not for a per-point δ that may shrink to zero as you move around. The mathematical difference is one re-ordering of quantifiers; the substantive difference between pointwise and uniform claims runs through analysis (and probability theory) like a fault line.

Uniform continuity strengthens pointwise continuity by demanding a single δ that works for all base points at once.

Definition 10. A function $f : A \to \mathbb{R}^m$ is *uniformly continuous* on A if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\boldsymbol{x}, \boldsymbol{y} \in A \text{ and } d(\boldsymbol{x}, \boldsymbol{y}) < \delta \implies d(f(\boldsymbol{x}), f(\boldsymbol{y})) < \epsilon.$$

The quantifier reads: $\forall \epsilon > 0 \, \exists \delta > 0 \, \forall \boldsymbol{x}, \boldsymbol{y} \in A$, whereas continuity reads $\forall \boldsymbol{x} \in A \, \forall \epsilon > 0 \, \exists \delta > 0 \, \forall \boldsymbol{y} \in A$. Pulling the $\forall \boldsymbol{x}$ inside the $\exists \delta$ is what makes the condition stronger; the same δ has to work everywhere on A.

Example 11 (Continuous but not uniformly continuous)**.** The function $f(x) = 1/x$ is continuous on $(0, \infty)$ but not uniformly continuous there. Near $x = 0$, the derivative $-1/x^2$ blows up, and the δ needed to keep f within ϵ shrinks to 0 as $x \to 0$. No single $\delta > 0$ works for all $x \in (0, \infty)$. (On any sub-interval $[a, \infty)$ with $a > 0$, however, f is uniformly continuous — the derivative is bounded there.)

Example 12 (Uniformly continuous)**.** Any function with bounded derivative is uniformly continuous (by the mean value theorem, $|f(x) - f(y)| \le M|x - y|$ for M a bound on $|f'|$, so $\delta = \epsilon/M$ works). In particular, $\sin x$ and $\cos x$ are uniformly continuous on $\mathbb{R}$, even though $\mathbb{R}$ is not bounded.

The compactness of the domain is the standard sufficient condition for uniform continuity to come for free.

Theorem 13 (Heine–Cantor)**.** *A continuous function on a compact subset of $\mathbb{R}^n$ is uniformly continuous.*

Proof. Suppose for contradiction that $f : K \to \mathbb{R}^m$ is continuous on the compact K but not uniformly continuous. Then there exists $\epsilon_0 > 0$ and sequences $(\boldsymbol{x}_k), (\boldsymbol{y}_k)$ in K with $d(\boldsymbol{x}_k, \boldsymbol{y}_k) < 1/k$ but $d(f(\boldsymbol{x}_k), f(\boldsymbol{y}_k)) \ge \epsilon_0$. By compactness, $(\boldsymbol{x}_k)$ has a convergent subsequence $\boldsymbol{x}_{k_j} \to \boldsymbol{x} \in K$. Since $d(\boldsymbol{x}_{k_j}, \boldsymbol{y}_{k_j}) \to 0$, also $\boldsymbol{y}_{k_j} \to \boldsymbol{x}$. By continuity of f (sequential characterization), both $f(\boldsymbol{x}_{k_j}) \to f(\boldsymbol{x})$ and $f(\boldsymbol{y}_{k_j}) \to f(\boldsymbol{x})$, so $d(f(\boldsymbol{x}_{k_j}), f(\boldsymbol{y}_{k_j})) \to 0$, contradicting $d(f(\boldsymbol{x}_{k_j}), f(\boldsymbol{y}_{k_j})) \ge \epsilon_0$. □

The proof is short, but the content is real: compactness rules out the kind of "runaway" behavior at the edges of the domain that lets δ shrink without bound.[2]

[2] The phrasing "compactness rules out runaway" is informal but accurate. Think of the failure cases. (i) $1/x$ on $(0, 1]$ fails uniform continuity because as $x \to 0$, the function rises arbitrarily fast — but the domain is not compact (it is missing the limit point 0), so continuity has no way to constrain behavior there. Closing the domain at 0 would have forced us to specify the value of f at 0, and any such specification would make f discontinuous, ruling out the example. (ii) $\sin(x^2)$ on $\mathbb{R}$ is continuous but not uniformly continuous: the oscillations get arbitrarily fast as $x \to \infty$, and $\mathbb{R}$ is not compact (it is unbounded). On any bounded sub-interval the function is uniformly continuous — compactness in the form "closed and bounded" would have ruled out the unboundedness. The Heine–Cantor theorem says that compact domains rule out both pathologies simultaneously, and the proof above shows it is the sequential-compactness form ("every sequence has a convergent subsequence in the set") that is doing the work — the contradiction comes from extracting subsequences whose endpoints would have to converge to a common limit. The general moral is one we will see again in optimization and fixed-point theory: compactness is the topological hypothesis that turns "it can fail at the edges" into "there are no edges to fail at."

In particular, on a closed bounded interval $[a, b]$ every continuous function is uniformly continuous, which is the form of the result one most often invokes (in the construction of the Riemann integral, in proofs about uniform convergence of polynomial approximations, and so on).

6 What's next

This handout closes the analysis cluster. Three directions extend it:

- *Optimization*, eventually: the EVT gives existence of an optimum on a compact domain, but the optimization-theoretic story — KKT conditions, Lagrange multipliers, dynamic programming, convex analysis — builds on top, and is the natural workhorse for political-economy modeling. *Berge's theorem of the maximum* (continuity of the value function and upper hemicontinuity of the argmax correspondence in parameters) is the basic comparative-statics tool that ties continuity to parameterized optimization.
- *Fixed-point theorems*, eventually: Brouwer's theorem (a continuous self-map of a compact convex set has a fixed point) and Kakutani's extension to upper hemicontinuous correspondences are the existence-proof workhorses for Nash equilibrium, general equilibrium, and other game-theoretic and economic settings. Both require continuity in essential ways.
- *Probability and measure* (next cluster): a continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ is automatically a Borel-measurable function, which is what makes pre-images of measurable sets behave well under integration. Continuous random variables, distribution functions, and convergence of integrals all build on the topology developed in this cluster.

For broader treatments at this level, see Rudin (1976) (chapter 4) or Abbott (2015) (chapter 4); for a treatment with explicit attention to the political-economy applications (compactness of strategy spaces, continuity of best-response correspondences, theorem of the maximum), see Ok (2007).

7 Exercises

Exercise 14. Show that $f(x) = x^2$ is continuous on $\mathbb{R}$ from the ϵ-δ definition. (Hint: $|x^2 - x_0^2| = |x - x_0| \cdot |x + x_0|$. The factor $|x + x_0|$ is bounded as long as x is restricted to a neighborhood of x_0, which is a feature, not a bug.)

Exercise 15. Prove the algebra-of-continuity sum rule directly from Theorem 2: if $f, g : A \to \mathbb{R}$ are continuous at $\boldsymbol{x}$, so is $f + g$.

Exercise 16. Show that the composition of two continuous functions is continuous: if $f : A \to B$ is continuous at $\boldsymbol{x}$ and $g : B \to C$ is continuous at $f(\boldsymbol{x})$, then $g \circ f$ is continuous at $\boldsymbol{x}$. Use either the ϵ-δ definition or the sequential characterization, your choice.

Exercise 17 (Continuity of a best-response function)**.** Consider a Cournot duopoly: two firms simultaneously choose quantities $q_1, q_2 \in [0, Q]$, the market price is $p(q_1 + q_2) = \max(0, A - q_1 - q_2)$ for some $A > 0$, and firm i's profit is $\pi_i(q_1, q_2) = q_i \cdot p(q_1 + q_2) - c \cdot q_i$ for marginal cost $c \in (0, A)$. Show that firm 1's best-response function $b_1(q_2) := \arg\max_{q_1 \in [0,Q]} \pi_1(q_1, q_2)$ is well-defined (the maximizer exists and is unique) and continuous in q_2. The continuity is what underwrites a Brouwer-fixed-point existence proof for the Cournot–Nash equilibrium, which we will not give here.

Exercise 18. Show that the image of a compact set under a continuous function is compact: if $K \subseteq \mathbb{R}^n$ is compact and $f : K \to \mathbb{R}^m$ is continuous, then $f(K) \subseteq \mathbb{R}^m$ is compact. (Hint: sequential characterization both sides — pick a sequence in $f(K)$, lift to a sequence in K, extract a subsequence, push forward.) Conclude EVT: in $\mathbb{R}$, a compact set's image is closed and bounded, so the supremum is attained.

Exercise 19 (Existence of a market-clearing price)**.** Continuing Example 7: suppose excess demand $z : [0, P] \to \mathbb{R}$ is continuous, $z(0) > 0$, and $z(P) < 0$. Apply the IVT directly to produce a market-clearing price. Now construct a discontinuous z for which no market-clearing price exists — e.g., a step function modeling minimum-quantity sellers — and explain in a sentence what economic feature the discontinuity captures.

Exercise 20. Let $f : [0, 1] \to \mathbb{R}$ be continuous with $f(0) = f(1)$. Show there exists $c \in [0, 1/2]$ with $f(c) = f(c + 1/2)$. (This is the "mountain climber" theorem: two climbers ascending opposite sides of a mountain with the same elevation profile must, at some moment, be at the same height. Apply the IVT to $g(x) := f(x) - f(x + 1/2)$.)

Exercise 21. Show $f(x) = 1/x$ is continuous on $(0, \infty)$ but not uniformly continuous. Identify the smallest $a > 0$ such that f *is* uniformly continuous on $[a, \infty)$ for every choice of $a > 0$ (the answer is that any $a > 0$ works, but you should articulate why).

Exercise 22 (Continuity in parameters)**.** A leader chooses $a \in [0, 1]$, then a follower chooses $b \in [0, 1]$ to maximize $u(a, b) := -(b - a^2)^2 - b^2/4$. Compute the follower's best response $b^*(a)$ explicitly, and show it is continuous in a. Now repeat with payoffs $u(a, b) := -(b - a^2)^2 \cdot \mathbf{1}_{\{b \neq 0\}} - 100 \cdot \mathbf{1}_{\{b=0\}}$ (which makes $b = 0$ uniquely terrible). Is the new $b^*(a)$ still continuous in a? What goes wrong, and which assumption of *Berge's theorem of the maximum* (the standard general result about continuity of the argmax in parameters; we will state it formally in a later handout) is being violated?

Exercise 23. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and define $g(x) := \sup_{|y-x| \leq 1} f(y)$. Show that g is well-defined (the supremum is attained) and continuous on $\mathbb{R}$. (Hint: the sup is over a compact set, so EVT applies pointwise; for continuity of g, use uniform continuity of f on each closed bounded interval.)

References

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