Choice under certainty

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1 Motivation

Order theory left preferences sitting in isolation: a relation on a set of alternatives, with axioms about reflexivity, transitivity, and completeness, and a question — when does there exist a real-valued utility representation — answered by Debreu. Applied work, however, does not observe preferences. It observes *choices*: a primary voter selects a candidate from a menu, an agenda setter shrinks a feasible set to its winning subset, a citizen sorts policy options into approve and reject. The theoretical framework that handles preferences and the empirical framework that handles choices have to be reconciled.

The reconciliation is a triangle. Three vertices: preferences (the order-theoretic object from the previous handouts), utilities (the real-valued functions representing them), and choice correspondences (the maps from menus to selections). Each pair of vertices is connected by a substantive theorem in each direction. Preferences are represented by utilities under Debreu's conditions. Utilities induce choices via $\arg\max$. Choices reveal preferences under the consistency conditions known as the weak axiom of revealed preference, the strong axiom, and the generalized axiom. The three theorems are not symmetric — each direction has its own conditions — but the handout's job is to make all three explicit and to mark the points where each can fail.

Throughout, we stay abstract. Alternative spaces are arbitrary sets, menus are arbitrary nonempty finite subsets, choice is a correspondence rather than a function, and the budget family is whatever the analyst happens to be able to observe. The Walrasian budget-set machinery from microeconomic theory specializes the framework, but it is not the only setting that matters. Voters do not face price-and-income parametrizations; agenda setters do not present budgets in $\mathbb{R}^n_+$. The abstract framework covers what the political-economy applications actually look like.

2 Preferences, briefly recapped

The order-theory handout established the basic vocabulary, and we will not redo it. A preference relation $\succsim$ on an alternative space X is a complete preorder: reflexive, transitive, and complete (any two alternatives are comparable, possibly via indifference). The strict part $\succ$ and indifference part $\sim$ are derived from $\succsim$ as in order theory §2. A utility function $u : X \to \mathbb{R}$ *represents* $\succsim$ if

$$x \succsim y \iff u(x) \geq u(y) \quad \text{for all } x, y \in X.$$

For finite X a representation always exists, by ranking the indifference classes; for infinite X the existence is conditional, with Debreu's separability condition (countable order-dense subset) the standard sufficient hypothesis. Lexicographic preferences on $[0, 1]^2$ exemplify the case where no representation exists.

Within this handout we treat the existence of u as background machinery, available when the alternative space is well-behaved and absent when it is not. The new work is on the choice side of the triangle: what kind of object choice is, and what consistency conditions on it correspond to its being driven by an underlying preference.

3 Choice functions and choice correspondences

Preferences are unobservable. Choices are observable. The translation between them starts by being precise about what "a choice" is. The applied setting suggests the answer immediately: an agent is presented with a menu of feasible alternatives, and the analyst records what the agent selects. The selection may be a single alternative, or it may be a tied set of alternatives the agent is willing to take indifferently. The agent's behavior across menus is what we want to study, and the right object is therefore not a single choice but a family of choices indexed by menu.

Definition 1. A *menu* (or *decision problem*) on X is a nonempty finite subset $A \subseteq X$. A *budget family* on X is a collection $\mathcal{B} \subseteq 2^X$ of menus from which the agent's choice is to be observed. The terminology is conventional; "decision problem" is the more common phrasing in psychology, "budget" the more common phrasing in economics.

Definition 2. A *choice correspondence* on $\mathcal{B}$ is a map $C : \mathcal{B} \to 2^X$ assigning to each $A \in \mathcal{B}$ a nonempty subset $C(A) \subseteq A$. A choice correspondence with $|C(A)| = 1$ for every A is a *choice function*.[1]

Example 3 (Primary voter facing a candidate menu)**.** A voter in a primary observes a menu of candidates and selects one. Across the primary cycle the menu changes as candidates enter and drop out, and what the voter selects from each menu is the empirical record of her choice correspondence. The natural budget family is the collection of menus the voter actually faces; the natural question is whether her choices are consistent with some underlying preference.

Example 4 (Agenda setter shrinking the motion set)**.** A committee chair restricts the motion set from a longer list to a shorter one. The committee's chosen motion from $\{a, b, c\}$ may differ from its chosen motion from any of the two-element sub-menus $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. The relation between the chair's restriction and the eventual choice is the agenda-manipulation phenomenon, and the question of whether the committee's choices are rationalizable by a single preference is the formal version of asking whether the chair's manipulation is doing real work.

Example 5 (Sen's $\{a, b, c\}$ vs. $\{a, b\}$ pattern)**.** A committee chooses a from $\{a, b, c\}$ but b from $\{a, b\}$. The pattern is structurally surprising: the alternative chosen from the larger menu is unchosen from the sub-menu, while a previously-unchosen alternative becomes chosen. We will revisit this in §5; for now it serves as a marker that not every choice correspondence is well-behaved.

[1] There is a real choice to be made about whether to require single-valuedness, and this handout does not. The case for the correspondence is twofold. First, a single-valued C silently assumes the agent's preference is antisymmetric (or strict), which collapses indifference to identity — two distinct alternatives the agent is genuinely indifferent between would have to be reduced to one of them by some external tie-breaking rule that has nothing to do with the agent's preference. The order-theory convention that preferences are complete preorders, not partial orders, is the principled reason for working with correspondences. Second, when an agent's choice is mediated by a strategy — a mixed strategy in game theory, a randomized voting rule in social choice — the support of the strategy is exactly the set of alternatives the agent is willing to play with positive probability, which is the natural reading of $C(A)$ when the agent is indifferent across the support. The correspondence framework is the cleaner home for both indifference and randomization. Single-valuedness can always be imposed later as an additional assumption when it is empirically warranted; pre-imposing it forecloses cases that show up in practice.

The structure of the budget family $\mathcal{B}$ matters for how informative a choice correspondence can be. If $\mathcal{B}$ contains all nonempty finite subsets, the analyst observes the agent across every conceivable menu and the data is maximal. If $\mathcal{B}$ contains only certain subsets — only the pairs, only triples, only the full X — the data is correspondingly less informative. The conditions in §4 and §5 below are sensitive to $\mathcal{B}$ being rich enough.

4 Rationalizability

Most of the choice correspondences a political-economy modeler thinks about have a particular shape. The agent has a fixed preference; she faces a menu; what she chooses is whatever sits on top of the preference inside the menu — the menu's $\succsim$-maxima, no more, no less. That shape is the cleanest one, and a great deal of the formal apparatus of choice theory is set up to characterize when an observed choice correspondence has it. The technical name is *rationalizability.*

Definition 6. A choice correspondence $C : \mathcal{B} \to 2^X$ is *rationalizable* if there exists a complete preorder $\succsim$ on X such that, for every $A \in \mathcal{B}$,

$$C(A) = \{x \in A : x \succsim y \text{ for every } y \in A\}.$$

The preorder $\succsim$ is then said to *rationalize* C.

The right-hand side is the set of $\succsim$-maxima of A — which, since $\succsim$ is complete and A is finite, is always nonempty. So rationalizable choice correspondences automatically have $C(A) \neq \emptyset$, which is built into Definition 2 from the start; the substantive content is that $C(A)$ is exactly the set of maxima of a single $\succsim$ that does not depend on A.

Proposition 7. *Suppose C is rationalizable on a budget family $\mathcal{B}$ that contains every two-element subset of X. Then the rationalizing preorder is uniquely determined by C on those pairs:*

$$x \succsim y \iff x \in C(\{x, y\}).$$

Proof. The right-hand side is the set of $\succsim$-maxima of $\{x, y\}$ — which is $\{x\}$ if $x \succ y$, $\{y\}$ if $y \succ x$, and $\{x, y\}$ if $x \sim y$. Reading off, $x \in C(\{x, y\})$ iff $x \succsim y$. □

The point of the proposition is that pair-wise choices contain enough information to recover $\succsim$ *when* rationalizability holds. The harder problem is to determine, from the data alone, whether C is rationalizable in the first place.

5 WARP, SARP, and Houthakker's theorem

The agent's choice is rationalizable iff her choices are consistent across menus in a precise sense. The two consistency conditions that do most of the work, and the resulting representation theorems, are the empirical workhorses of revealed-preference theory. The vocabulary and the basic axioms go back to Samuelson (1938), with the abstract finite-menu development due to Arrow (1959) and Sen (1971); Houthakker (1950) is the source of the strongest representation result.

We start by extracting from C a relation on X that records what the agent's choices reveal.

Definition 8. The *revealed weak preference* $\succsim^*$ on X is defined by

$$x \succsim^* y \iff \exists\, A \in \mathcal{B} \text{ with } x \in C(A) \text{ and } y \in A.$$

The *revealed strict preference* $\succ^*$ is defined by $x \succ^* y \iff x \succsim^* y$ and not $y \succsim^* x$.

The reading: $x \succsim^* y$ when, in some observed menu containing y, the agent chose x; $x \succ^* y$ when she chose x over y but never the other way around. Note that $\succsim^*$ need not be reflexive or transitive a priori — it is just a relation read off from the data.

Definition 9 (WARP)**.** C satisfies the *weak axiom of revealed preference* (WARP) if, for every $x, y \in X$ with $x \succsim^* y$ and $y \succsim^* x$, and for every $A \in \mathcal{B}$ with $x, y \in A$,

$$x \in C(A) \iff y \in C(A).$$

In words: if both x and y have been observed weakly preferred to each other, then in every menu containing both, they are chosen together (or rejected together). WARP is the cross-menu consistency condition: indifference revealed in one menu must propagate to every other menu where both alternatives are available.

The Arrow–Sen representation theorem says WARP is exactly the characterization of rationalizability when the budget family is rich enough.

Theorem 10 (Arrow–Sen)**.** *Let $\mathcal{B}$ contain every two- and three-element subset of X. Then C is rationalizable by a complete preorder iff C satisfies WARP.*

Proof sketch. **($\Rightarrow$).** If $\succsim$ rationalizes C, suppose $x \succsim^* y$ and $y \succsim^* x$. From $x \succsim^* y$, there is a menu A with $x \in C(A)$ and $y \in A$, so $x \succsim y$ (by maximality of x). Symmetrically $y \succsim x$. So $x \sim y$. In any menu containing both, the maxima are an upper-closed set under $\sim$, so $x \in C(A) \iff y \in C(A)$. **($\Leftarrow$).** Define $\succsim$ to be the relation $\succsim^*$. Reflexivity follows since each alternative is chosen from its singleton menu (every singleton is in $\mathcal{B}$ by hypothesis on two-element subsets). Completeness: for any x, y, the menu $\{x, y\} \in \mathcal{B}$ has nonempty $C(\{x, y\})$, which is either $\{x\}$, $\{y\}$, or $\{x, y\}$, giving in each case $x \succsim^* y$ or $y \succsim^* x$. Transitivity is the substantive step: if $x \succsim^* y$ and $y \succsim^* z$, consider the menu $\{x, y, z\} \in \mathcal{B}$ and use WARP on the three two-element sub-menus to argue $x \in C(\{x, y, z\})$, hence $x \succsim^* z$. The full argument requires patient bookkeeping; see Sen (1971, Theorem 1) for a complete proof. To verify $C(A) = \{x \in A : x \succsim y \text{ for all } y \in A\}$ for every $A \in \mathcal{B}$ is then a check using the same WARP-on-pairs argument applied to A. □

WARP captures rationalizability when the budget family is rich on small menus. When the budget family is sparse — containing, say, only a few menus, none of them small — WARP can be trivially satisfied without the choices being rationalizable. The strong axiom is the consistency condition that handles that case.

Definition 11 (SARP)**.** C satisfies the *strong axiom of revealed preference* (SARP) if there is no sequence $x_1, x_2, \ldots, x_n$ in X with $x_i \succ^* x_{i+1}$ for $i = 1, \ldots, n-1$ and $x_n \succ^* x_1$. (No revealed-strict-preference cycles.)

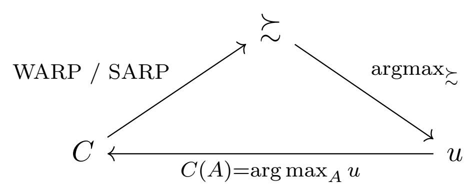
Theorem 12 (Houthakker)**.** *On any budget family $\mathcal{B}$, C is rationalizable by a complete preorder iff C satisfies SARP.*

We omit the proof; see Houthakker (1950) and Richter (1966).[2]

WARP and SARP capture rationalizability under different conditions on $\mathcal{B}$. The empirical content is the same: they ask whether the agent's observed choices are consistent with the existence of *some* preference she is maximizing. Failure of either is failure of that hypothesis, and the next section's synthesis displays the role each axiom plays in closing the loop.

6 The preference–utility–choice triangle

We have three objects on the table now — preferences, utility, and choice — and three theorems connecting them. It is worth pausing to draw the picture, because the three theorems are subtly different in character and depend on different conditions, and a working political economist who reaches for this material in the middle of a paper needs to know which theorem applies in which direction. The picture is a triangle. Each vertex is one of the three objects, each edge is a theorem running between them, and the canonical constructions in each direction are:



Going around the triangle: a preference relation admits a utility representation under Debreu's conditions (the missing third arrow $\succsim\to u$, deferred to order theory §6). The utility representation induces a choice correspondence by maximization over each menu. The choice correspondence in turn determines a preference relation via WARP or SARP, completing the loop.

The composition is sensitive: each arrow has its own conditions, and the loop closes only when all three sets of conditions are met.

- $\succsim\to u$ *(utility representation)* requires X to be well-behaved (finite, or a topological space with a countable order-dense subset and a continuous preference). Lexicographic preferences on $[0,1]^2$ are the canonical case where the arrow fails: the preference exists, but no u represents it.

- $u \to C$ *(induced choice via argmax)* requires u to attain its maximum on each menu. For finite menus this is automatic; for compact menus it follows from continuity of u; for non-compact menus or discontinuous u the arrow can fail and the induced "choice" is empty.

[2] The relationship among WARP, SARP, and the *generalized* axiom of revealed preference (GARP) deserves a note. WARP rules out two-cycles in the revealed-strict-preference relation; SARP rules out all finite cycles. On budget families containing all small subsets (specifically, all two- and three-element subsets), WARP and SARP are equivalent — the proof of the Arrow–Sen theorem extends to longer cycles by induction on the cycle length. Outside that case, SARP is strictly stronger. GARP weakens SARP slightly to allow indifference within cycles (any cycle with at least one $\sim$-step is allowed); the divergence between SARP and GARP does not show up on abstract finite menus, but does on Walrasian budget sets, where consumer choice can have indifference structure that SARP rules out and GARP does not. The mainstream microeconomic-theory tradition, working in the Walrasian setting, treats GARP as the substantive axiom; see Mas-Colell, Whinston, and Green (1995, Ch. 3) and Varian (1982) for the budget-set development. The political-economy applications (candidate menus, motion sets, agenda choices) sit naturally in the abstract framework where WARP/SARP do most of the work, which is why this handout uses them.

- $C \to \succsim$ *(rationalization via revealed preference)* requires the budget family $\mathcal{B}$ to be rich enough and the consistency axiom (WARP for rich $\mathcal{B}$, SARP in general) to hold. The Arrow–Sen and Houthakker theorems are exactly the statements of those conditions.

The triangle closes when each arrow's conditions are satisfied. The point of having all three is that the failures are real phenomena, each diagnostic of something interesting in the data: a non-representable preference is a structural feature of the alternative space (lex preferences are not pathological — they are an honest model of certain real attitudes); an empty argmax is a model-misspecification flag (the analyst has built a u on a domain where the maximization problem doesn't make sense); a non-rationalizable C is empirical evidence that the agent is not maximizing any single preference at all.

The third failure mode is the politically interesting one. When a population's choices fail WARP, the population cannot be modeled as a single preference-maximizing agent. The Condorcet cycle exhibits this exactly: three voters with cyclic majority preferences induce a population-level choice correspondence that violates WARP, and treating the population as a representative agent runs into the impossibility we know from social choice theory. The categorical-style version of this observation lives in the model-theory-and-modeling handout's treatment of Arrow's theorem; here it is the operational version that the diagnostic axioms catch.

7 What's next

The next handout extends the alternative space from X to $\Delta(X)$, the set of probability distributions over X, and asks for the analogue of utility representation for preferences over lotteries. The answer is the von Neumann–Morgenstern expected-utility theorem, with the independence axiom doing the load-bearing work; the Allais paradox shows independence failing in real choice. The handout after that drops the assumption that probabilities are given and introduces the Anscombe–Aumann and Savage frameworks for choice under ambiguity.

For graduate-level treatments at this handout's level of abstraction, Kreps (1988) is the standard accessible reference, and Mas-Colell, Whinston, and Green (1995, Ch. 1) is the canonical microeconomic-theory presentation. Austen-Smith and Banks (1999) is the political-science-native counterpart, working through preferences, utility, and choice with PE applications throughout; Austen-Smith and Banks (2005) continues into the strategic and institutional setting. Sen (1971) is the source for the abstract-finite-menu axiom system used here.

8 Exercises

Exercise 13. Verify that for any choice correspondence C, the revealed weak preference $\succsim^*$ is reflexive on the alternatives that appear in some chosen set. (That is: if $x \in C(A)$ for some A, then $x \succsim^* x$.)

Exercise 14. Build the preference relation that rationalizes the following choice correspondence on $X = \{x, y, z\}$: $C(\{x, y\}) = \{x\}$, $C(\{y, z\}) = \{y\}$, $C(\{x, z\}) = \{x\}$, $C(\{x, y, z\}) = \{x\}$. Verify your relation is a complete preorder and that it indeed produces these chosen sets.

Exercise 15. (*Sen's α.*) Show that any rationalizable C satisfies the *contraction property*: if $x \in C(A)$ and $x \in B \subseteq A$ with $B \in \mathcal{B}$, then $x \in C(B)$. (Hint: a $\succsim$-maximum of A that is in B is also a $\succsim$-maximum of B.)

Exercise 16. (*Sen's β.*) Show that any rationalizable C satisfies the *expansion property*: if $x, y \in C(A)$ and $A \subseteq B$ with $A, B \in \mathcal{B}$, then $x \in C(B) \iff y \in C(B)$. (The two together — α and β — are equivalent to WARP on rich enough $\mathcal{B}$; this is the standard Sen-style decomposition of WARP.)

Exercise 17. A three-person committee has the following pattern of pairwise majority choices: $C(\{a, b\}) = \{a\}$, $C(\{b, c\}) = \{b\}$, $C(\{a, c\}) = \{c\}$, $C(\{a, b, c\}) = \{b\}$. (a) Compute the revealed weak preference $\succsim^*$. (b) Is C rationalizable? (c) Discuss: by selecting which sub-menu to put on the floor, could an agenda setter induce different final choices, and what does the answer say about strategic agenda manipulation?

Exercise 18. A primary voter is observed making the following choices across overlapping menus: from $\{A, B, C, D\}$ she chooses C; from $\{A, B, C\}$ she chooses A; from $\{B, C, D\}$ she chooses B. (a) Does the voter's choice violate WARP? (b) Is it rationalizable? (c) In one or two sentences, discuss what kind of structure the voter's behavior might have if she is not maximizing a single preference.

Exercise 19. (*Sen's α/β failure.*) A committee chooses a from $\{a, b, c\}$ but b from $\{a, b\}$. (a) Show that this pattern violates Sen's α. (b) Argue that no preference relation rationalizes the pattern. (c) Discuss: why might a real committee exhibit such a pattern? (Hint: the menu shapes the strategic context. When c is on the floor, voting for a over b may be safe; when c leaves, the safety is gone and b becomes the modal choice.)

Exercise 20. Show that SARP implies WARP. Then exhibit a budget family $\mathcal{B}$ and a choice correspondence C on $\mathcal{B}$ that satisfies WARP but violates SARP. (Hint: take $\mathcal{B}$ to contain only menus of size four or more. WARP can be vacuous when no two-element menus are observed.)

Exercise 21. (*Condorcet cycles as population-level WARP failure.*) Three voters have preferences $a \succ b \succ c$, $b \succ c \succ a$, $c \succ a \succ b$. The population-level choice from a menu is determined by simple majority pairwise voting (with the round-robin tiebreaker that the chosen alternative is the one with the most pairwise wins). (a) Compute the population-level choice from each two-element menu and from the full menu $\{a, b, c\}$. (b) Compute the revealed weak preference $\succsim^*$ at the population level. (c) Show that the population's choice violates WARP. (d) Discuss in one or two sentences what this says about the limitations of treating a population as if it were a single preference-driven agent — the formal version of Arrow's impossibility, in revealed-preference clothing.

Exercise 22. Suppose X is a finite alternative space and $\mathcal{B}$ contains every nonempty subset. Show that the rationalizing preference $\succsim$, when it exists, is unique — so a finite, rich-enough budget family pins down the entire preference relation from the choice data alone. (This is the cleanest case in which choice fully determines preference; outside it, the determination is partial.)

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