Categories and functors

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1 Motivation

The reader has by now met several settings in which a class of objects came packaged with a class of structure-preserving maps between them: posets and order-preserving functions; metric spaces and continuous functions; measurable spaces and measurable functions; sets and functions, even. Each setting has its own theorems, but the patterns rhyme. Composition of structure-preserving maps is structure-preserving. Identity maps preserve everything. The right notion of "sameness" is the existence of a structure-preserving map with a structure-preserving inverse. Category theory is the language for that recurring pattern.

The payoff for a political-economy reader is mainly cognitive. Once one has the language, the recognition that a coalition lattice is a thin category, that an aggregation rule is a functor, that two mechanism designs are the same iff the categories of their type-and-allocation data are equivalent — each of these lets one transfer arguments instead of re-proving them. A second payoff is for reading: structural arguments (Tarski's fixed-point theorem, Berge's theorem of the maximum, equivalence of mechanism designs, the categorical packaging of stochastic processes) are usually category-flavored under the hood, even when the surface vocabulary is different. The model-theoretic style of formal political economy — a model is a structure, a class of models is a class of structures (Suppes, 2002; the previous handout) — becomes more legible once one sees that "structure plus structure-preserving map" is the categorical idiom.

We will not need many results. The working theorem of this cluster is the Yoneda lemma in the next handout. What we need here is fluency with the basic vocabulary — categories, functors, the universal-property style of definition — and that is what this handout supplies.

2 Categories

A category is the lightest sensible structure in which composition makes sense and identities are built in. The data are deliberately spare, but they capture what most "kinds of mathematical objects" have in common: a class of things, a class of maps between them, and a sensible composition law.

Definition 1. A *category* $\mathcal{C}$ consists of:

- a class $\mathrm{Ob}(\mathcal{C})$ of *objects*;
- for each pair of objects A, B, a set $\mathrm{Hom}_{\mathcal{C}}(A, B)$ of *morphisms* (or *arrows*) from A to B;
- for each triple A, B, C, a *composition* operation

$$\circ : \mathrm{Hom}_{\mathcal{C}}(B, C) \times \mathrm{Hom}_{\mathcal{C}}(A, B) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(A, C);$$

- for each object A, an *identity morphism* $\mathrm{id}_A \in \mathrm{Hom}_{\mathcal{C}}(A, A)$;

satisfying:

- (associativity) $h \circ (g \circ f) = (h \circ g) \circ f$ whenever the composites are defined;
- (identity) $\mathrm{id}_B \circ f = f = f \circ \mathrm{id}_A$ for every $f : A \to B$.

A morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ has a *source* A and a *target* B; we write $f : A \to B$ or $A \xrightarrow{f} B$. Composition only makes sense when the source of one matches the target of the other — it is a partial operation, not a global one.[1]

The first thing one wants is a stockpile of examples, both because the abstraction lives or dies on them and because the right examples make the right theorems suggest themselves.

Example 2 (**Set**)**.** Objects: sets. Morphisms $A \to B$: functions $A \to B$. Composition: function composition. Identity id_A: the identity function. The associativity of function composition and the unit laws of identity functions give the axioms for free.

Example 3 (**Pos**)**.** Objects: posets. Morphisms: order-preserving functions. Composition is well-defined because the composition of two order-preserving maps is order-preserving (order-theory §6 Exercise). Identity: the identity function is order-preserving.

Example 4 (**Top**, **Meas**)**.** Objects: topological spaces / measurable spaces. Morphisms: continuous functions / measurable functions. The closure of these classes under composition and identity is exactly what makes "continuous" and "measurable" the right structural conditions for the respective settings.

Example 5 (A preorder is a thin category)**.** Let $(A, \preceq)$ be a preorder. Define a category $\mathcal{A}$ with $\mathrm{Ob}(\mathcal{A}) = A$, and

$$\mathrm{Hom}_{\mathcal{A}}(a, b) = \begin{cases} \{*\} & \text{if } a \preceq b, \\ \emptyset & \text{otherwise.} \end{cases}$$

Composition is forced (each Hom-set has at most one element), and the identity $\mathrm{id}_a : a \to a$ exists because $a \preceq a$. Associativity and the unit laws are vacuous. So a preorder *is* a category — one with at most one morphism between any two objects, also called a *thin* category. Conversely, every thin category arises this way: the objects form a preorder under the relation "there exists a morphism from a to b." Preorders and thin categories are the same thing dressed up differently.

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[1] There is a foundational subtlety here that turns out to be the analogue, in category theory, of the size pathologies that drove the move from naive to axiomatic set theory. The cardinality-and-infinity handout's discussion of Russell's paradox showed the cost of unrestricted comprehension on the set side: there is no *set* of all sets, on pain of contradiction. The categorical analogue is the same warning rephrased. A category is *small* if both its objects and its morphisms form sets, and *locally small* if each $\mathrm{Hom}(A, B)$ is a set, even when the class of objects may be a proper class. The category **Set** in Example 2 below is locally small but not small — its objects, the sets, form a proper class. The category **Cat** of *small* categories (Exercise 25) is itself also locally small but not small, for the same reason: small categories form a proper class. The discipline that prevents Russell-style paradoxes in the categorical setting is exactly the size-stratification one knows from set theory — one never speaks of the category of *all* categories *including itself*, only of the locally-small-but-not-small category of all *small* categories. A more thoroughgoing fix posits a tower of *Grothendieck universes* $U_0 \in U_1 \in \cdots$, each closed under set-theoretic operations, so that $\mathbf{Set}_U$ becomes an object of some larger-universe $\mathbf{Set}_{U'}$; this is the standard setup in homological algebra and algebraic geometry. The pattern is the same in both settings: Russell's paradox is not avoided by collecting more carefully; it is avoided by stratifying so that no collection ever contains itself, whether the collection is a set or a category. We will not dwell on size issues, but it is worth knowing they are there. The standard reference is Mac Lane (1998, Ch. 1).

Example 6 (A monoid is a one-object category)**.** Let $(M, \cdot, e)$ be a monoid. Define a category with one object $\star$, with $\mathrm{Hom}(\star, \star) = M$, composition = monoid multiplication, identity = e. The associativity and identity axioms of a category are exactly the monoid axioms. A *group* is the special case where every morphism is invertible. Conversely, any one-object category gives rise to a monoid, and any one-object category in which every morphism is invertible (a *groupoid* on one object) gives rise to a group.

Examples 5 and 6 are worth pausing on. They show that "category" generalizes both ordered things (preorders) and arrow things (monoids). The breadth is a feature: a single notion of structure underwrites both "a is below b" and "g then f."[2]

A construction worth recording up front is the opposite category, which lets every result be stated in two flavors at once.

Definition 7. The *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$, with $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$, and composition reversed: $g \circ_{\mathcal{C}^{\mathrm{op}}} f := f \circ_{\mathcal{C}} g$. Reversing the arrows of $\mathcal{C}^{\mathrm{op}}$ recovers $\mathcal{C}$ on the nose.

The op-construction is mostly a notational device — it lets us state a theorem about $\mathcal{C}$ once and apply it to $\mathcal{C}^{\mathrm{op}}$ to get the dual statement, instead of reproving things twice.

3 Iso, mono, epi

In **Set**, three notions of well-behaved function — injective, surjective, bijective — capture three different desiderata. In a general category we have no elements to lean on, but the cancellation properties that make those notions work can be stated purely in terms of composition. The result is the trio mono / epi / iso, which agrees with injective / surjective / bijective in **Set** but generalizes sensibly elsewhere — and where it doesn't agree, it tells us something interesting about the category.

Definition 8. A morphism $f : A \to B$ in $\mathcal{C}$ is:

- an *isomorphism* (or *iso*) if there exists $g : B \to A$ with $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$;
- a *monomorphism* (or *mono*) if it is left-cancellable: $f \circ h_1 = f \circ h_2 \Rightarrow h_1 = h_2$ for any $h_1, h_2 : C \to A$;
- an *epimorphism* (or *epi*) if it is right-cancellable: $h_1 \circ f = h_2 \circ f \Rightarrow h_1 = h_2$ for any $h_1, h_2 : B \to C$.

Two objects are *isomorphic*, written $A \cong B$, if there is an iso between them.

In **Set** the three notions land where one expects.

Proposition 9. *In* **Set**, *a function $f : A \to B$ is mono iff injective, epi iff surjective, iso iff bijective.*

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[2] The *preorder + monoid* pair is the simplest version of a duality that recurs throughout the subject. A preorder collapses every Hom-set to size at most one, and so all the action lives in the objects. A one-object category collapses every object to a single one, and so all the action lives in the morphisms. The general case — many objects, many morphisms between any pair — combines both styles of information, which is why the abstraction is genuinely more general than either piece alone.

Proof. **Mono iff injective:** if f is injective and $f \circ h_1 = f \circ h_2$, then $h_1(c) = h_2(c)$ for every c, so $h_1 = h_2$. Conversely, if f is not injective, take $a \neq a'$ with $f(a) = f(a')$ and $C = \{*\}$, $h_1, h_2 : \{*\} \to A$ picking out a, a'; then $f \circ h_1 = f \circ h_2$ but $h_1 \neq h_2$. **Epi iff surjective:** similar, using $C = \{0, 1\}$ and the indicator of the image. **Iso iff bijective:** a function has a two-sided inverse iff it is bijective. □

In other categories the correspondence can break.[3] The mismatch matters: a structural argument that conflates the three is silently using set-theoretic content, and the category-theoretic statement keeps the notions separate so we can see what is really going on.

The deeper moral is structural, and worth pausing on. "One-to-one" presupposes that we can talk about elements: that an object has an inside, populated by points, with $f(a) \neq f(a')$ certifying the property by inspection. Mono asks for no such thing. It only asks that f behave well under composition. In settings where the objects are not transparent containers of further units — thin categories (where "elements of a" is not a sensible question), functor categories and presheaf categories, the structural-relations categories that arise in international relations theory and organizational economics where the units are aggregate actors treated as black boxes — "one-to-one" is not even askable, but "mono" is. So one-to-one is the more restrictive of the two notions, in the sense that it presupposes more about what the objects are made of. Wherever both make sense, the two coincide; wherever only one makes sense, it is mono. The same holds for epi versus onto. The trio mono / epi / iso is a structural vocabulary that survives the loss of internal-element data; the injective / surjective / bijective trio does not. Whenever the analytical move on the table is to bracket the internal composition of a unit and reason about its external relations — which is exactly the move IR scholars make when they treat states as units, and one of the moves political economists make when they reason about coalitions, parties, or organizations as primitive entities — the categorical mono / epi vocabulary is the one that actually carries the argument.

4 Universal properties: initial, terminal, products, coproducts

The next step is to characterize objects by what morphisms go in and out of them, rather than by what they are made of. This is the *universal-property* style, and it is what makes category theory feel structurally different from set-theoretic mathematics. Where set theory builds objects out of elements, the universal-property approach describes objects by their relations to others — the kind of relational characterization a political theorist uses when she defines a coalition by what it can do (form a winning bloc) rather than by who is in it. The simplest universal properties are those involving exactly one morphism in or out: initial and terminal objects.

Definition 10. An object $0 \in \mathcal{C}$ is *initial* if for every A there is exactly one morphism $0 \to A$. An object $1 \in \mathcal{C}$ is *terminal* if for every A there is exactly one morphism $A \to 1$.

Example 11 (Initial and terminal objects in standard categories).

- **Set**: $\emptyset$ is initial (there is exactly one function from $\emptyset$ into any set, the empty function); any singleton $\{*\}$ is terminal (the unique constant function).

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[3] The classic mismatches are diagnostic. In **Top**, the inclusion $\mathbb{Q} \hookrightarrow \mathbb{R}$ is both mono and epi — mono because it is injective, and epi because a continuous function out of $\mathbb{R}$ is determined by its restriction to a dense subset, so the right-cancellation condition holds even though the inclusion is not surjective. In the category of rings, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is both mono and epi for analogous reasons. So in either category, "iso = mono + epi" is false. The structural moral is that mono and epi together are the right generalization of "injective and surjective," but neither one is a generalization of "bijective"; iso is. Theorems that elsewhere conflate the three are quietly using set-theoretic content, and the categorical statement keeps the notions separate.

- **Pos**, **Top**, **Meas**: same.
- $(A, \preceq)$ as a thin category: an initial object is a $\preceq$-minimum; a terminal object is a $\preceq$-maximum. (The Hasse-diagram bottom and top, when they exist.)
- A monoid as a one-object category: the unique object is both initial and terminal, since the data of a one-object category does not distinguish them.

Initial and terminal objects, when they exist, are unique up to a unique iso: any two terminal objects each receive a unique morphism from the other, and the composites are forced (by uniqueness) to be the identities. This "unique up to unique iso" pattern recurs and is why one can speak of "the" terminal object without ambiguity.

The next two definitions exhibit the universal-property style at full strength.

Definition 12. A *product* of A and B in $\mathcal{C}$ is an object $A \times B$ together with morphisms $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ such that for every object C and every pair of morphisms $f : C \to A$, $g : C \to B$, there is a unique morphism $\langle f, g\rangle : C \to A \times B$ with $\pi_A \circ \langle f, g\rangle = f$ and $\pi_B \circ \langle f, g\rangle = g$.

The data is laid out by the universal-property diagram:

$$\begin{tikzcd}
 & C \arrow[ld, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[rd, "g"] & \\
A & A \times B \arrow[l, "\pi_A"] \arrow[r, "\pi_B"'] & B
\end{tikzcd}$$

The dashed arrow is the unique one whose existence the definition asserts. Read the diagram aloud: "every f, g as drawn factor uniquely through $A \times B$." That is the product, and the universal-property style of definition more generally.

Example 13 (Products).

- **Set**: the cartesian product, with the standard projections.
- **Pos**: the cartesian product of underlying sets, with the *product order* $(a, b) \preceq (a', b') \iff a \preceq_A a'$ and $b \preceq_B b'$ (cf. the order-theory exercise on products of posets).
- In a preorder $(A, \preceq)$ as a thin category: the product of a and b is the *meet* (greatest lower bound) $a \wedge b$, when it exists. Reason: a morphism $C \to a \wedge b$ in the thin-category sense is exactly an element c with $c \preceq a$ and $c \preceq b$, and that is the universal property of the meet. So binary meets in a poset are categorical products in the thin category.

Definition 14. A *coproduct* of A and B in $\mathcal{C}$ is an object $A + B$ together with morphisms $\iota_A : A \to A + B$ and $\iota_B : B \to A + B$ such that for every C and every pair $f : A \to C$, $g : B \to C$, there is a unique $[f, g] : A + B \to C$ with $[f, g] \circ \iota_A = f$ and $[f, g] \circ \iota_B = g$.

Coproducts are the dual of products: reverse all the arrows in the product diagram. In **Set**, the coproduct is the disjoint union $A \sqcup B$ from naive set theory §5; in **Pos**, it is the disjoint union of posets with elements from different summands incomparable; and in a poset as a thin category, the coproduct is the *join* $a \vee b$. The lattice operations of order-theory §5 are the categorical product and coproduct in the thin-category translation, full stop.

5 Functors

Once a category is the right notion of "structure," the right notion of "map between structures" is one that respects the data: send objects to objects, morphisms to morphisms, and respect composition and identity. That is a functor.

Definition 15. Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *functor* $F : \mathcal{C} \to \mathcal{D}$ assigns:

- to each object $A \in \mathcal{C}$ an object $F(A) \in \mathcal{D}$;
- to each morphism $f : A \to B$ in $\mathcal{C}$ a morphism $F(f) : F(A) \to F(B)$ in $\mathcal{D}$;

satisfying:

- $F(g \circ f) = F(g) \circ F(f)$ for composable f, g;
- $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$ for every A.

A *contravariant functor* $F : \mathcal{C} \to \mathcal{D}$ reverses arrows: $F(f : A \to B) : F(B) \to F(A)$, with $F(g \circ f) = F(f) \circ F(g)$. Equivalently, a contravariant functor $\mathcal{C} \to \mathcal{D}$ is a (covariant) functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

The two axioms can be summarized in slogan form: a functor is a "structure-preserving map of compositional data." The associativity built into the categories on either side does not need to be assumed of F separately — it is inherited.

Example 16 (Identity functor)**.** $\mathrm{id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}$ sends every object and morphism to itself.

Example 17 (Forgetful functors)**.** The functor $U : \mathbf{Pos} \to \mathbf{Set}$ sends a poset $(A, \preceq)$ to its underlying set A and an order-preserving function to the underlying function. The order is forgotten. Analogously $\mathbf{Top} \to \mathbf{Set}$, $\mathbf{Meas} \to \mathbf{Set}$, $\mathbf{Grp} \to \mathbf{Set}$, and so on. Forgetful functors are the most banal-looking of all, but they are also one of the most useful: many constructions are characterized by adjunction relations involving a forgetful functor on one side. (Adjunctions are beyond this handout's scope, but Mac Lane (1998) or Riehl (2017) treat them at length.)

Example 18 (Power-set functor)**.** $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ sends a set A to its power set $\mathcal{P}(A)$, and a function $f : A \to B$ to the *image map* $\mathcal{P}(f) : \mathcal{P}(A) \to \mathcal{P}(B)$, $S \mapsto f(S)$. Verify functoriality: $\mathcal{P}(g \circ f)(S) = (g \circ f)(S) = g(f(S)) = \mathcal{P}(g)(\mathcal{P}(f)(S))$. There is also a contravariant version, $\mathcal{P}^* : \mathbf{Set} \to \mathbf{Set}^{\mathrm{op}}$, sending f to the *preimage map* $S \mapsto f^{-1}(S)$. The same set $\mathcal{P}(A)$ supports two different functors — one covariant via images, one contravariant via preimages.

Example 19 (Hom-functors)**.** For an object A in a locally small $\mathcal{C}$, define $\mathrm{Hom}(A, -) : \mathcal{C} \to \mathbf{Set}$ on objects by $B \mapsto \mathrm{Hom}_{\mathcal{C}}(A, B)$, and on a morphism $g : B \to B'$ by post-composition: $\mathrm{Hom}(A, g)(f) = g \circ f$ for $f \in \mathrm{Hom}(A, B)$. Verify functoriality: composition becomes composition, identity becomes identity. The contravariant version $\mathrm{Hom}(-, A) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ sends $h : C \to C'$ to the pre-composition map $\mathrm{Hom}(C', A) \to \mathrm{Hom}(C, A)$, $f \mapsto f \circ h$. These two Hom-functors are the central characters of the next handout.

Example 20 (A small PE-flavored example)**.** Let a finite legislature $N = \{1, \ldots, n\}$ have its coalition lattice $(\mathcal{P}(N), \subseteq)$, viewed as a thin category $\mathcal{N}$. A *Boolean voting rule* is a function $w : \mathcal{P}(N) \to \{0, 1\}$ that is monotone (a coalition that wins, wins after gaining a member). Treating $\{0, 1\}$ as the two-element thin category $\mathbf{2}$ (with $0 \leq 1$), w is precisely a functor $\mathcal{N} \to \mathbf{2}$. The functor axioms are vacuous (composition is forced in thin categories); what they encode is monotonicity. The structural framing makes monotonicity look like exactly what it is: "preservation of the order" is "preservation of the categorical structure" once order and category are seen as the same thing. The minimal-winning-coalitions exercise from order theory is then the question, "what coalitions are mapped to 1 by w but every proper sub-coalition to 0?" — a question about the support of the functor, not about the rule itself.

The categorical packaging in Example 20 does not make the model do new work; it makes the model's structure visible.

6 Equivalence of categories

When are two categories "the same"? Set-theoretic equality is too strong: the category of finite-dimensional real vector spaces and the category of natural-number-indexed matrix algebras are not the same on the nose, but every working linear algebraist treats them as such. *Equivalence of categories* is the right notion. It captures structural sameness without demanding object-level identity.

The vocabulary builds on three properties of functors.

Definition 21. A functor $F : \mathcal{C} \to \mathcal{D}$ is:

- *faithful* if for each pair $A, B \in \mathcal{C}$, the induced map $F : \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ is injective;
- *full* if that map is surjective;
- *essentially surjective* if every $D \in \mathcal{D}$ is isomorphic to $F(A)$ for some $A \in \mathcal{C}$.

A functor is *fully faithful* if it is both full and faithful.

The forgetful functor $U : \mathbf{Pos} \to \mathbf{Set}$ (Example 17) is faithful (two distinct order-preserving maps between the same posets remain distinct as functions) but not full (a function between underlying sets need not be order-preserving) and is essentially surjective (every set is the underlying set of some poset — e.g., the discrete one with no nontrivial comparisons).

Definition 22. Categories $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* if there exist functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with isomorphisms $G \circ F \cong \mathrm{id}_{\mathcal{C}}$ and $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ *at the level of functors*. The functor F is then called an *equivalence of categories*.

The phrase "isomorphism at the level of functors" is doing real work, and we cannot make it precise until the next handout.[4] What can be stated cleanly already is the following structural characterization, which says the right-looking "F is fully faithful and hits everything up to iso" diagnostic captures equivalence.

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[4] The point: equivalence does *not* require $G \circ F = \mathrm{id}_{\mathcal{C}}$ on the nose, only that the two functors $G \circ F$ and $\mathrm{id}_{\mathcal{C}}$ be isomorphic as functors. Functors are themselves objects of a (functor) category, and isomorphism in that category is

Theorem 23 (Equivalence characterization)**.** *A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence iff it is fully faithful and essentially surjective.*

We defer the proof to the next handout, where the natural-transformation language is on hand. The point of stating it here is to record what equivalence *looks like* from F alone — it is detectable on Hom-sets and on objects-up-to-iso, without any reference to the partner functor G.

Example 24 (Preorders and thin categories are the same)**.** The construction in Example 5 extends to a functor between the category of preorders and the (full sub-)category of thin categories: send a preorder to the thin category it generates, and a morphism (an order-preserving map) to the corresponding functor. The construction is fully faithful and essentially surjective by Example 5's analysis. So preorders and thin categories are equivalent as categories: the same structural content in two presentations.

7 What's next

Two strands extend this handout:

- *Natural transformations and the Yoneda lemma* (next handout). Functors between two fixed categories form a category of their own, with morphisms called natural transformations. The headline theorem of the cluster is the Yoneda lemma: an object A in $\mathcal{C}$ is determined — up to a unique iso — by the functor $\mathrm{Hom}(A, -)$. The slogan "an object is what it is by virtue of its maps in" becomes precise. The next handout proves it.

- *Heavier categorical machinery*, eventually. Limits and colimits (universal cones, of which products and coproducts are the two-object cases), adjoint functors (the relationship between the forgetful $U : \mathbf{Pos} \to \mathbf{Set}$ and its left adjoint, and the dozens of other forgetful-free pairs), monads (a packaging of effectful or stateful computation), and the tools to recognize when a functor preserves all the structure one cares about. These are beyond the present cluster. The interested reader may look ahead at Mac Lane (1998, Chs. 3–5) or, for a politer exposition, Awodey (2010) or Riehl (2017).

For category theory itself, the standard graduate references are Mac Lane (1998) and Awodey (2010). Riehl (2017) is a more modern take with broader audience and is freely available online; Leinster (2014) is short, friendly, and an excellent first read.

8 Exercises

Exercise 25. Show that the composition of two functors is again a functor. (Verify both axioms in turn.) Conclude that there is a category **Cat** whose objects are small categories and whose morphisms are functors.

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what we want. The morphisms in the functor category are *natural transformations*, and that is the next handout's first task. Equivalence-as-defined-now is shorthand for "functors-naturally-isomorphic" that becomes precise once natural transformations are in hand. The looser-than-equality condition is precisely what makes equivalence the working notion — demanding $G \circ F = \mathrm{id}_{\mathcal{C}}$ would force every object of $\mathcal{C}$ to be the image of some object of $\mathcal{D}$ and back, which is rarely true even for categories everyone agrees are "the same."

Exercise 26. Show that every isomorphism in any category is both mono and epi. Give a one-sentence reason why the converse can fail outside **Set**. (Cf. Footnote on $\mathbb{Q} \hookrightarrow \mathbb{R}$ in **Top**.)

Exercise 27. In **Pos**, characterize the iso, mono, and epi morphisms in terms of the underlying functions. (Mono: injective. Epi: surjective. Iso: order-isomorphism — i.e., a bijection that is order-preserving with order-preserving inverse. Show that an order-preserving bijection need not be an iso, by exhibiting one whose inverse is not order-preserving.)

Exercise 28. A thin category has *no* "one-to-one" or "onto" notions in any element-level sense, since there are no elements to inspect. Show that, nonetheless, in a thin category every morphism is both mono and epi. (Hint: any equation $f \circ h_1 = f \circ h_2$ between morphisms with the same source and target is automatically true in a thin category, because thin categories have at most one morphism between any two objects.) Compare the classical-set-theory question "is $\preceq$-restriction a one-to-one map between coalition lattices" with its categorical reformulation "is the corresponding functor mono"; only the second has a coherent answer in the structural setting.

Exercise 29. Let $N = \{1, 2, 3, 4\}$ be a four-member legislature, with coalitions ordered by inclusion — the thin category $\mathcal{N}$. (a) What is the product of $\{1, 2\}$ and $\{2, 3\}$ in $\mathcal{N}$? The coproduct? (b) Identify the initial and terminal objects. (c) Verify that the simple-majority winning predicate $w(\sigma) = \mathbf{1}[|\sigma| \geq 3]$ is a functor $\mathcal{N} \to \mathbf{2}$ (Example 20). What changes if the rule is weighted, say $w(\sigma) = \mathbf{1}[\sum_{i\in\sigma} q_i \geq 1]$ with weights q_i summing to 2?

Exercise 30. Let V_1 and V_2 be voter-type spaces, viewed as objects of **Meas**, and let $r : V_1 \to V_2$ be a measurable refinement function (each V_1-type has a unique V_2-type, but several V_1-types may share one). Show that r is mono in **Meas** iff r is injective on points (so that no information is lost in the refinement direction), and epi iff r has dense image with respect to the measurable structure of V_2. Why does "every V_2-type is the refinement of some V_1-type" translate to a categorical statement about r?

Exercise 31. Prove that the forgetful functor $U : \mathbf{Pos} \to \mathbf{Set}$ is faithful but not full. (One sentence each.) Identify a pair of posets and a function between their underlying sets that is not the underlying function of any order-preserving map between the posets.

Exercise 32. Compute $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, B)$ for an arbitrary set B (Example 19 with $A = \{*\}$). Conclude that the functor $\mathrm{Hom}_{\mathbf{Set}}(\{*\}, -) : \mathbf{Set} \to \mathbf{Set}$ is, in a sense to be made precise in the next handout, the identity functor on **Set**.

Exercise 33. Let $\mathcal{G}$ be the category whose objects are finite normal-form games on a fixed set $N = \{1, \ldots, n\}$ of players, and whose morphisms are payoff-preserving *relabelings* of strategies (a morphism $G \to G'$ consists of bijections $\sigma_i : S_i^G \to S_i^{G'}$ for each player such that the payoff functions are intertwined). Show that every morphism in $\mathcal{G}$ is iso. (Such a category is called a *groupoid*.) Why is this the right structural rendering of "two games are the same up to relabeling of strategies"?

Exercise 34. A *constant functor* $\Delta_D : \mathcal{C} \to \mathcal{D}$ sends every object of $\mathcal{C}$ to a fixed $D \in \mathcal{D}$, and every morphism to id_D. (a) Verify functoriality. (b) When is Δ_D faithful? Full? Essentially surjective? (c) Use the constant functor to show that $\mathcal{D}$ has a terminal object iff there is a fully faithful functor **Set** $\to \mathcal{D}$ sending $\{*\}$ to that terminal object. (Hint: a fully faithful functor on a one-object category is the same data as picking out an object together with its automorphism monoid.)

Exercise 35. Let $\mathcal{A}$ and $\mathcal{B}$ be two preorders viewed as thin categories (Example 5). (a) Show that a functor $F : \mathcal{A} \to \mathcal{B}$ is exactly an order-preserving function between the underlying preorders. (b)

Show that any such F is automatically faithful. (c) Show that F is full iff $F(a) \preceq F(a')$ implies $a \preceq a'$ (the converse of order-preservation). (d) State, in plain order-theoretic terms, what it means for two preorders, viewed as thin categories, to be equivalent in the sense of Definition 22. (Answer: equivalent posets-of-equivalence-classes under the symmetric part.)

References

Awodey, Steve (2010). *Category Theory.* 2nd ed. Oxford: Oxford University Press.

Leinster, Tom (2014). *Basic Category Theory.* Cambridge: Cambridge University Press.

Mac Lane, Saunders (1998). *Categories for the Working Mathematician.* 2nd ed. New York: Springer.

Riehl, Emily (2017). *Category Theory in Context.* Mineola, NY: Dover Publications.

Suppes, Patrick (2002). *Representation and Invariance of Scientific Structures.* Stanford: CSLI Publications.