Cardinality and infinity

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1 Motivation

How big is a set? For finite sets the question is easy: count the elements. For infinite sets the question is less obvious, and the answer turns out to be unexpectedly rich. There are not just "finite" and "infinite" but a whole hierarchy of infinities, distinguished from one another by whether one can be put into bijection with the other. The basic distinction — countable versus uncountable — already does most of the practical work, but the deeper theorems (Cantor's, Schröder–Bernstein, the role of the continuum hypothesis) are worth meeting at least once because they organize how much later mathematics is structured.

For political economy, the practical translation is "discrete versus continuous." A finite or countably infinite set of voters, alternatives, types, or states is discrete; an uncountable set — an interval of policy positions, a continuum of agent types — is continuous. The two settings call for different machinery: combinatorics and discrete probability on the discrete side; topology, continuity, and measure theory on the continuous side. The cardinality material in this handout is the bridge between the two and the entry point to the analysis cluster that follows.

2 Equinumerosity and comparing sizes

How do you compare the sizes of two infinite sets? For finite sets the answer is trivial — count both, compare the numbers — but counting itself requires a finite stopping point, and the sets we want to compare don't have one. Cantor's basic move, which turns out to be exactly the right one, is to compare sizes by exhibiting *bijections* rather than by counting. Two sets are "the same size" if there is a way to pair off their elements one-to-one; one set is "at least as large as" another if there is an injection one way.

Definition 1. Two sets A and B are *equinumerous* (or have the same *cardinality*), written $|A| = |B|$ or $A \approx B$, if there exists a bijection $f : A \to B$.

Proposition 2. *Equinumerosity is an equivalence relation.*

Proof. Reflexive: the identity map $A \to A$ is a bijection. Symmetric: if $f : A \to B$ is a bijection, $f^{-1} : B \to A$ is a bijection. Transitive: the composition of bijections is a bijection. □

Definition 3. $|A| \leq |B|$ means there exists an injection $f : A \to B$. We write $|A| < |B|$ when $|A| \leq |B|$ and not $|A| = |B|$.

The relation $\leq$ on cardinalities is reflexive (identity injection) and transitive (composition of injections). Antisymmetry is the content of a basic but non-trivial theorem.

Theorem 4 (Schröder–Bernstein)**.** *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.*

The theorem is non-trivial because the two given injections can fail to be bijective, and the conclusion is that some bijection nevertheless exists. We omit the (constructive) proof; the reader can find it in any text on set theory.[1]

The practical use of Schröder–Bernstein is to prove $|A| = |B|$ by exhibiting two injections rather than constructing a single bijection directly. We will use it shortly.

3 Countable sets

The countably infinite sets are the smallest infinite ones — small enough that one can, in principle, list out all the elements as $a_0, a_1, a_2, \ldots$, even if the list never ends. For applied work the distinction matters: a model with a countable type space (binary signals, integer-valued vote totals, the set of all finite voter histories) supports discrete probability and combinatorial reasoning, while uncountable type spaces (continuous signals, real-valued ideal points, the unit interval of policy positions) require the topological and measure-theoretic machinery of later handouts. The first task of this section is to show that several sets one might reflexively call "much larger than $\mathbb{N}$" — the integers, the rationals, $\mathbb{N} \times \mathbb{N}$ — are nonetheless countable.

The smallest infinite cardinality is $|\mathbb{N}|$, denoted $\aleph_0$ ("aleph-null"). A set with this cardinality is called *countably infinite*; finite sets and countably infinite sets together are *countable*.

Example 5 (The integers are countable)**.** The map $f : \mathbb{N} \to \mathbb{Z}$ defined by $f(0) = 0$, $f(2k-1) = k$, $f(2k) = -k$ for $k \geq 1$ is a bijection. So $|\mathbb{Z}| = |\mathbb{N}|$.

Example 6 ($\mathbb{N}\times\mathbb{N}$ is countable)**.** List the pairs in a zigzag: $(0,0), (1,0), (0,1), (2,0), (1,1), (0,2), (3,0), (2,1), \ldots$, walking diagonally through the grid. This produces a bijection $\mathbb{N} \to \mathbb{N} \times \mathbb{N}$, so $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$.

Example 7 (The rationals are countable)**.** Each $q \in \mathbb{Q}^+$ can be written uniquely as p/q with $\gcd(p, q) = 1$ and $q > 0$, giving an injection $\mathbb{Q}^+ \to \mathbb{N} \times \mathbb{N}$. Combining with the integer-style trick to extend to negative rationals gives an injection $\mathbb{Q} \to \mathbb{N} \times \mathbb{N}$. By the previous example, $|\mathbb{Q}| \leq |\mathbb{N}|$. Conversely, $\mathbb{N} \subset \mathbb{Q}$ gives $|\mathbb{N}| \leq |\mathbb{Q}|$. By Schröder–Bernstein, $|\mathbb{Q}| = |\mathbb{N}|$.

The closure properties of countable sets:

Proposition 8. *A countable union of countable sets is countable. The Cartesian product of finitely many countable sets is countable.*

Proof. For the union: list each A_i (which is countable) as $a_{i,1}, a_{i,2}, \ldots$, and walk diagonally through the doubly indexed array, exactly as in the $\mathbb{N} \times \mathbb{N}$ example, skipping repeated elements. For the product: induct on the number of factors using $A \times B \approx \mathbb{N} \times \mathbb{N} \approx \mathbb{N}$ when A, B are countable. □

This is enough to handle most countably infinite spaces that arise in political economy: the set of all finite-length voter histories, the set of all polynomials with rational coefficients, the set of all coalitions when the legislator set itself is countably infinite.

[1] Cardinal trichotomy — "for any sets A and B, exactly one of $|A| < |B|$, $|A| = |B|$, $|B| < |A|$ holds" — requires the axiom of choice. Schröder–Bernstein, in contrast, does not, which is part of why it is the workhorse for proving equinumerosity in practice. Without choice, it is consistent (with ZF) that there exist incomparable cardinalities.

4 Uncountable sets

If countability stretched indefinitely — if every infinite set could be put in bijection with $\mathbb{N}$ — the cardinality story would end here. It does not. Cantor's theorem is that the real numbers are not countable.

Theorem 9 (Cantor)**.** *$\mathbb{R}$ is uncountable, i.e., $|\mathbb{N}| < |\mathbb{R}|$.*

Proof. We give Cantor's diagonal argument for $|[0,1]| > |\mathbb{N}|$, from which $|\mathbb{R}| \geq |[0,1]| > |\mathbb{N}|$ follows. Suppose for contradiction that $[0,1]$ were countable, so we could list its elements as $x_1, x_2, x_3, \ldots$. Write each x_n as a decimal expansion (using the convention that we never use a tail of all 9s, so each real has a unique representation):

$$x_n = 0.\, d_{n,1}\, d_{n,2}\, d_{n,3}\, \ldots$$

Now construct a new number $y \in [0,1]$ by choosing the nth digit e_n of y to differ from the diagonal digit $d_{n,n}$ — say, $e_n = 1$ if $d_{n,n} \neq 1$, and $e_n = 2$ otherwise (avoiding 0 and 9 to dodge expansion ambiguities). Then y differs from each x_n in the nth decimal place, so $y \neq x_n$ for any n. But $y \in [0,1]$, contradicting the assumption that the list was complete. □

The argument is laid out as a table, with the constructed y disagreeing with the listed x_n on the diagonal:

n		decimal expansion					
1	:	0.	$\mathbf{d_{1,1}}$	$d_{1,2}$	$d_{1,3}$	$d_{1,4}$	$\cdots$
2	:	0.	$d_{2,1}$	$\mathbf{d_{2,2}}$	$d_{2,3}$	$d_{2,4}$	$\cdots$
3	:	0.	$d_{3,1}$	$d_{3,2}$	$\mathbf{d_{3,3}}$	$d_{3,4}$	$\cdots$
4	:	0.	$d_{4,1}$	$d_{4,2}$	$d_{4,3}$	$\mathbf{d_{4,4}}$	$\cdots$
y	:	0.	e_1	e_2	e_3	e_4	$\cdots$

The diagonal element $d_{n,n}$ is the one e_n is constructed to differ from. By design, y is on no row.

Example 10 ($|[0,1]| = |\mathbb{R}|$)**.** The map $\mathbb{R} \to (0,1)$ given by $x \mapsto (1 + e^{-x})^{-1}$ (the logistic function) is a bijection, so $|\mathbb{R}| = |(0,1)|$. The injection $(0,1) \hookrightarrow [0,1]$ is obvious; the reverse injection $[0,1] \hookrightarrow (0,1)$ is, e.g., $x \mapsto (x+1)/3$. By Schröder–Bernstein, $|[0,1]| = |\mathbb{R}|$. So all real intervals of positive length are equinumerous with $\mathbb{R}$.

We write $\mathfrak{c} := |\mathbb{R}|$ for the cardinality of the continuum.

5 Cantor's theorem and the hierarchy of cardinals

The diagonal argument from the previous section was about $\mathbb{R}$ specifically, but the structure of the argument doesn't depend on the reals at all. The same move applies to any set: the power set is strictly larger than the set itself. This is Cantor's theorem, and it has the dramatic consequence that there is no largest cardinality — given any infinity, applying the theorem produces a strictly bigger one, and iterating yields an unbounded hierarchy of infinities. The first non-trivial calculation in this hierarchy connects two cardinalities we have already met: $|\mathcal{P}(\mathbb{N})|$ turns out to equal $|\mathbb{R}|$.

Theorem 11 (Cantor)**.** *For any set A, $|A| < |\mathcal{P}(A)|$.*

Proof. $|A| \leq |\mathcal{P}(A)|$ via the injection $a \mapsto \{a\}$. To show strict inequality, suppose for contradiction that $f : A \to \mathcal{P}(A)$ is a surjection. Define

$$D = \{a \in A : a \notin f(a)\}.$$

Since f is surjective, $D = f(a_0)$ for some $a_0 \in A$. Now ask whether $a_0 \in D$. If $a_0 \in D$, then by the defining property of D, $a_0 \notin f(a_0) = D$ — contradiction. If $a_0 \notin D$, then $a_0 \notin f(a_0)$, so by the defining property $a_0 \in D$ — contradiction. □

The argument is the same diagonal as Cantor's argument for $\mathbb{R}$, in the form that makes it visible.

The hierarchy starts at $|\mathbb{N}|$ and climbs without end:

$$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots,$$

with each level strictly bigger than the last. The first calculation that pins down a level uses an alphabet trick:

Proposition 12. $|\mathcal{P}(\mathbb{N})| = \mathfrak{c}$.

Proof. A subset $S \subseteq \mathbb{N}$ corresponds to a binary sequence $(b_0, b_1, b_2, \ldots) \in \{0,1\}^{\mathbb{N}}$ via $b_n = 1$ if $n \in S$ and 0 otherwise. So $|\mathcal{P}(\mathbb{N})| = |\{0,1\}^{\mathbb{N}}|$. The latter is in bijection with the binary expansions of numbers in $[0,1]$ (modulo the dyadic-rational ambiguity, which only changes things by countably many points and so doesn't affect cardinality). Hence $|\mathcal{P}(\mathbb{N})| = |[0,1]| = \mathfrak{c}$. □

A natural question is whether there are cardinalities *between* $|\mathbb{N}|$ and $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$. The answer is famously not what one might hope for.[2]

6 Discrete versus continuous; order density

For applied work, the cardinality trichotomy mostly translates as "discrete" versus "continuous." A discrete set is countable: voters in an election, states in a finite-state Markov model, strategies in a game with finitely many actions, the integer-valued vote totals at the close of a poll. A continuous set is typically uncountable: the unit interval of policy positions, $\mathbb{R}^n$ as an outcome space, $[0,1]$ as a continuum of types in a Bayesian game. The distinction matters not just for counting but for what mathematics applies. Discrete probability, combinatorics, and discrete optimization live on countable sets; integration, continuity, topology, and measure theory live on uncountable ones.

[2]The *continuum hypothesis* (CH) asserts that there is no set S with $|\mathbb{N}| < |S| < |\mathbb{R}|$ — equivalently, that the cardinality of the continuum is the next cardinality after $\aleph_0$, traditionally written $\aleph_1$. Cantor formulated the hypothesis in 1878 and could neither prove nor disprove it; it sat as the first problem on Hilbert's famous 1900 list. The resolution came in two stages. Gödel (1940) showed that CH cannot be *disproved* from the standard ZFC axioms: there is a model of ZFC in which CH holds. Cohen (1963) showed that CH cannot be *proved* from ZFC either: there is a model of ZFC in which CH fails. So CH is *independent* of ZFC — it is neither a theorem nor a refutation, but a separate axiom one can either accept or reject. The independence result extends to the generalized continuum hypothesis ($|\mathcal{P}(\aleph_\alpha)| = \aleph_{\alpha+1}$ for every ordinal α). For applied work the moral is: the precise cardinalities of the higher levels of the cumulative hierarchy are mostly not pinned down by standard set theory, and most concrete questions in analysis or probability theory turn out to be insensitive to the answer.

The order-theoretic side has a more interesting story. Recall from the order theory handout that lexicographic preferences on $[0,1]^2$ admit no real-valued utility representation, because $([0,1]^2, \preceq^L)$ has no countable order-dense subset. The cardinality angle on this is direct: $[0,1]^2$ is uncountable in a way that $\mathbb{R}$ alone is not — the lex order has uncountably many "layers" (one for each value of the first coordinate), and a countable order-dense subset cannot reach into all of them. Whereas $(\mathbb{R}, \leq)$ *does* have a countable order-dense subset (namely $\mathbb{Q}$), and that is why preferences over a single continuous variable admit a utility representation.

The cardinality of $\mathbb{R}$ is large enough to model essentially any continuous space we will encounter in PE. The cardinality of $\mathcal{P}(\mathbb{R})$ — one strict step up the hierarchy — is large enough to be problematic.[3]

7 What's next

Two strands extend this handout:

- *Real analysis.* The least upper bound property of $\mathbb{R}$ — order-theoretic in flavor, set-theoretic in cardinality content — is the entry point to the analysis cluster: *sequences and limits*, *open and closed sets*, *continuity.* (See HANDOUTS.md for the cluster shape.)
- *Probability and measure.* Once we are working with uncountable spaces, the question of which subsets to assign probabilities to becomes pressing, and the apparatus of σ-algebras and measures is the answer. Probability theory in any modern PE-relevant form runs on this machinery: probability spaces, random variables, expectations, conditional expectations, convergence theorems. (Separate cluster, eventually.)

For broader treatment of the set-theoretic side, see Halmos (1960); for a deeper development of the order-and-cardinality interaction, Davey and Priestley (2002).

8 Exercises

Exercise 13. Show that the set of finite sequences over a fixed countable alphabet Σ, namely $\bigcup_{n\geq 0} \Sigma^n$, is countable. Conclude: in an election with countably many candidates, the set of all finite-length voter histories is countable; in a Markov decision process with countably many states, the set of all finite-length state trajectories is countable.

[3]When the alternative space or outcome space is uncountable, the next question is what to do with all of its subsets. The power set $\mathcal{P}(\mathbb{R})$ is too large to be useful directly: there is no way to assign a "length" or a "probability" to every subset of $\mathbb{R}$ in a way that respects countable additivity and translation invariance (the existence of *Vitali sets*, which use the axiom of choice, kills this; relatedly, the Banach–Tarski paradox shows that $\mathcal{P}(\mathbb{R}^3)$ is too rich to admit a finitely-additive isometry-invariant measure). The standard repair is to work with a *σ-algebra* — a sub-collection of $\mathcal{P}(\mathbb{R})$ closed under countable unions, complements, and containing the empty set — rather than the full power set. The Borel σ-algebra (the smallest σ-algebra containing all open sets) and the Lebesgue σ-algebra (Borel + null sets) are the standard choices. Cardinality lurks behind the structural fact: $|\mathcal{P}(\mathbb{R})| = 2^{\mathfrak{c}}$ is strictly larger than $\mathfrak{c}$, by Cantor's theorem, and the "extra room" is exactly where the pathological non-measurable sets live. The probability-and-measure handout takes this up properly; here the takeaway is that working with all subsets of an uncountable space is not in general a sensible thing to want.

Exercise 14. Show that the set of polynomials in one variable with rational coefficients is countable. Conclude that the set of *algebraic numbers* (real numbers that are roots of nonzero rational-coefficient polynomials) is countable, and hence that the set of *transcendental* real numbers (those that are not algebraic) is uncountable. (You do not need to exhibit a specific transcendental number.)

Exercise 15. Show that $|[0,1] \times [0,1]| = |[0,1]|$. (Hint: interleave the digits of two decimal expansions to produce a single one. Be careful at the boundary cases involving infinite trailing 9s.)

Exercise 16. Use Schröder–Bernstein to show that $|\mathbb{R}| = |\mathbb{R}^n|$ for every positive integer n. (You can build on the previous exercise.) Conclude that an n-dimensional continuous policy space $\mathbb{R}^n$ has the same cardinality as a one-dimensional one — a structurally surprising fact, since dimension obviously matters in many other respects (Pareto-incomparability, lex preferences, geometric structure).

Exercise 17. Show that the set of all functions $\mathbb{N} \to \{0,1\}$ is uncountable; in bijection with $\mathcal{P}(\mathbb{N})$, this is also the set of all coalitions of a countably infinite legislator set. Show further that the set of all functions $\mathbb{N} \to \mathbb{N}$ has cardinality $\mathfrak{c}$ — the same cardinality as the set of all sequences of integer-valued vote totals across countably many elections.

Exercise 18. Adapt Cantor's diagonal argument to show that for any infinite set A, $|\mathcal{P}(A)| > |A|$. (This is what was claimed in §5; spell out the argument explicitly for an arbitrary infinite A.)

Exercise 19. Discrete vs. continuous in PE. Suppose a model has a finite set of voters V with $|V| = n$, each voting for one of $|C| = m$ candidates. How many possible vote totals are there, and how many possible voting outcomes (i.e., functions $V \to C$)? Now suppose voters cast a vote in $[0,1]$ (a continuous score for a single candidate). What is the cardinality of the set of possible voting outcomes?

Exercise 20. Show that any two countable dense linear orders without endpoints (e.g., $(\mathbb{Q}, \leq)$ and $(\mathbb{Q} \cap (0,1), \leq)$) are order-isomorphic. (This is Cantor's theorem on countable dense linear orders. Hint: build the isomorphism by "back-and-forth.")

Exercise 21. Recall that lexicographic preferences on $[0,1]^2$ admit no real-valued utility representation. Suppose instead the underlying space is the countable set $(\mathbb{Q} \cap [0,1])^2$, with the induced lexicographic order. Does this restricted space admit a real-valued utility representation? (Hint: count the alternative space; recall the finite/countable case from the order theory handout.)

Exercise 22. Explain in your own words why $\mathcal{P}(\mathbb{R})$ is "too large" for measure theory — that is, why one cannot consistently assign a σ-additive translation-invariant probability to every subset of $\mathbb{R}$. (You do not need to construct a Vitali set; argue informally that some such pathology must exist on cardinality grounds, given Cantor's theorem.)

References

Davey, B. A. and H. A. Priestley (2002). *Introduction to Lattices and Order*. 2nd ed. Cambridge: Cambridge University Press.

Halmos, Paul R. (1960). *Naive Set Theory*. Princeton: Van Nostrand.